A Streetcar Named (among other things) Proof

From Storytelling to Geometry, via Poetry and Rhetoric

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APOSTOLOS DOXIADIS

1. The Genesis of Proof, from Miracle to Process

Even after various car lines were closed, their route names continued for a while on the roll signs carried on the streetcars. Motormen could often be induced to "turn the crank" and display an obsolete route name while they were taking a break at the end of a run, while a tourist snapped a no-longer-possible picture. With the popularity of the Tennessee Williams play and movie "A Streetcar Named Desire", it is no wonder that "Desire" was a popular choice for such posed pictures. But it was not the only choice.

—H. George Friedman Jr., *New Orleans Streetcar Album*

The naming of a New Orleans streetcar line "Desire" is a clear case of synecdoche, the calling of the whole after a part. "Desire," after Desire Street, was the name of a station in a line in the center of the city whose other stations included Elysian Streets, Cemeteries, Canal, Royal. Some of these offered alternative names: "Desire Line" was also occasionally called "Canal Line" or "Royal Line," and the motorman would turn the crank to display *those* names when required. This varying synecdochic naming of one and the same line by several of its stations provides a great—though admittedly unusual—metaphor for the story of the early days of logical demonstration, the process mathematicians usually refer to as *proof*.

The emergence in the archaic and classical Greek world of systematic rationality, a methodology of which mathematical proof is generally considered the highest peak, has been idealized in a dated topos of intellectual history as "the Greek Miracle." But though newer approaches examine the discovery of proof in a more historically informed light, there still remains at their center the idea of a discontinuity, of the emergence of rationality as a radical break with the past. This sense of a totally new practice is particularly strong in the historiography of the emergence of mathematical proof. In a recent authoritative study we read, "The origin of Greek mathematics could have been a *sudden explosion* of knowledge" (Netz 1999a, 273; my italics). The available material evidence certainly points to this discontinuity: first there is no proof, then—suddenly—there is. Of course, this impression is partly a result of our ignorance. Sadly, we have no extant texts of Greek mathematics in the form in which it was written before the end of the fourth century BCE, and any attempts to retrieve this style depend on texts written centuries later (Netz 2004).

The absence of early texts is not the only obstacle, however. Coupled with it is the fact that, until very recently, historians of mathematics worked almost exclusively in an *internalist* mode, studying the progress of mathematical concepts, techniques, and results solely from a viewpoint interior to the discipline, as if the mathematicians of old only spoke with, read, or were otherwise influenced by other mathematicians: mathematics begat mathematics. This orientation occasionally drove the discipline toward a Whig-historical attitude, whereby earlier work was read in the context of later progress. Well-known instances of this mentality are Thomas L.

Heath's readings of certain parts of classical geometry in the language of trigonometry and B. L. Van der Waerden's notion of the Greeks doing "geometric algebra."

An exclusively inwardly-focused investigation of a discipline is at an obvious disadvantage when one tries to address its beginning: by virtue of its being a beginning, there is nothing "inside" the discipline that could have acted as an antecedent cause or influence to it. Happily, however, the dogma of pure internalism in the history of mathematics has subsided in recent decades. The externalist viewpoint now prevalent in natural science history studies, a viewpoint based on the obvious premise that scientists do not work in a cultural vacuum, is also gaining ground in the history of mathematics. In the past few years, a number of scholars, most prominently Reviel Netz and Serafina Cuomo, have made important progress in applying a partly externalist approach—remember, good externalism is partial externalism—to the study of Greek mathematics.

Despite important recent advances, however, there still remains a significant lacuna in the history of Greek mathematics, in the origin story of deductive proof. Part of the reason for this is the burden of the central metaphor of "birth" or "genesis." For though modern scholarship has moved away from the thaumaturgical, the view of a sudden emergence of proof still dominates our thinking. It is in the place of these metaphors that I want to propose the streetcar-named-Desire metaphor. This has two advantages over "birth" or "genesis." First, it helps us see the invention of proof not as event but as process. In fact, instead of thinking of a single new methodology—a "station," as it were—called "proof," as formalized in the books of Aristotle's *Organon* and demonstrated in full use in Euclid's *Elements*, we can think of "proof" as a synecdoche for a "Proof Line", a long, developing process. As G. E. R. Lloyd writes, even "allowing for Aristotle's own originality, the concept of rigorous proof did not exactly spring, fully armed, like some Athena, from his head" (1990, 77). And second, the streetcar-named-Desire metaphor avoids the distortion of giving the particular station we call "proof" an ascendancy over earlier ones on the same line, a distortion that pushes the intellectual history of Greece in a teleologic direction, leading some scholars to read it as a course "toward rationality." But this need not be so: other intellectual motormen, approaching ancient history and culture with different agendas, name this same line after one of its earlier stations, branding it variously, instead of the Proof Line, the Rhetoric Line, or even—to show that this can be so is the main thesis of this essay—the Storytelling or the Poetry Line. It is crucial, then, to realize we are talking of stations on one and the same line.

The ideal landscape for this line to move in was partly revealed by the work of Gemet (1981) and Vernant (1984) and further developed by Vidal-Naquet (with Vernant 1988), Detienne (1999), and Loraux (2006). The gist of their approach is that the impetus for the development of the bundle of practices we call logic lay in the needs and opportunities provided by the new, participatory political institutions of city-states in the Greek world, in the sixth and fifth centuries BCE. The greatest advance in the investigation of systematic rationality in this direction, however, has been made by Lloyd, who, in his studies of early philosophy and science (1966, 1979, 1990), has gone a long way toward showing that it is not "just that there is a general sense in which Greek philosophy reflects the circumstances of Greek political experience: rather the latter is at least one important formative influence on the styles of debate that are typical of the former" (1990, 142). It should be noted, however, that it is more correct to take many of the arguments of these scholars as applying, more generally, to the various forms of participatory decision making by large groups of citizens known to have existed in archaic and classical Greece, rather than insisting on what we now call Greek *democracy*, as such, as the facilitating environment.[1]

It should be noted that the ideas of the so-called "French school" seem better equipped to explain the rationality of natural philosophers, such as Hippocrates of Kos, or philosophers, like Aristotle, than of mathematicians such as Autolycus or Euclid, and has not been applied to the history of mathematics except in its most general and thus not particularly useful sense. This is

not surprising: the "reason-from-democracy" thesis has at its center the art of rhetoric, an art crucial to both Hippocratic medicine and Aristotelian investigations. But rhetoric is an art working within the ambiguities of everyday language and probabilistic premises, whereas mathematics builds on absolute certainties, using a restricted and clearly defined language. Obviously rhetoric, an art of persuasion, also uses a form of proof. But this form, many scholars believe, is totally different from that of its mathematical namesake. My only objection to that assertion concerns the word *totally*. That rhetorical proof is not the same as mathematical proof is obvious—that they are unrelated is preposterous. Yet this denial of their essential similarity (*not* identity), ultimately derived from Platonic prejudice setting rhetoric at the antipodes of logic, has blocked the investigations of rhetorical practice from becoming essential to the cognitive history of logico-deductive proof.

In the view I adopt here, the two practices have a large area of overlap, as shown in simple diagrammatic form in figure 10.1. When the older, internalist historians of mathematics mentioned rhetoric at all, they did so while placing themselves squarely in the right-hand side of figure 10.1, casting their glance leftward only to highlight the *differences* of mathematics from rhetoric. However, we shall begin on the opposite side, first examining rhetoric in order to understand the similarities of the two practices, similarities it would take a gross leap of faith to consider accidental. After all, though early forms of rhetoric have older roots, both practices blossom at about the same time, in the latter part of the fifth century and on into the fourth century BCE, in circles of intellectually sophisticated citizens, living in a handful of Greek poleis, that is, in the Greek adversarial cultural context, which promoted "the ambition to secure a demonstration that would silence the opposition once and for all" (Lloyd 2002, 66). Identifying similarities between any two contemporaneous practices A and B begs the question of influence: did A beget B, did B beget A, or are both the offspring of a common ancestor, C? Based on the available textual evidence, in the case of Greek rhetoric (A) and mathematics (B) I would opt for a combination of the first and the third answers. However, as most of my arguments here are structural, they are really independent of the question of influence, and thus stay mostly clear of it.[2]

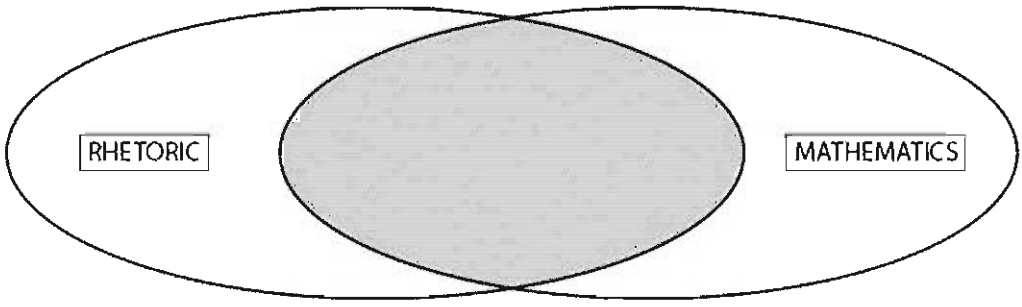

Figure 10.1. A diagrammatic depiction of the relationship of rhetoric and mathematics.

I shall begin my investigation slightly outside the area mapped in figure 10.1, on the left side. For before examining how certain rhetorical concepts, methods, and patterns were instrumental to mathematical proof, I want to examine out how they entered the domain of rhetoric in the first place: to say that mathematics took a large part of its logico-deductive, apodictic tools from rhetoric but leave the origin of these tools in rhetoric unexplained would simply relocate rather than answer the original question.

I shall use the streetcar-named-Desire metaphor to guide us through various cognitive modes, or stations, in historical development, beginning with the first of these, narrative. From there we move on to poetic storytelling, then to rhetoric, and finally to our ulterior goal, proof. This course is not simply linear, though, as depicted in figure 10.2.

Figure 10.2. A linear depiction of the historical sequences of cultural practices leading to proof.

Because the word *proof* describes two distinct, though related, practices, in rhetoric[3] and mathematics, the course mapped in figure 10.2 should be replaced by that in figure 10.3, which suggests a more adequate account of historical progress in the late Archaic and the Classical Ages. The solid black arrows denote direct influence, the dotted indirect, while the gray arrows indicate exclusively domain-specific influences from the two domains in which the need for such proofs was born. Though there will be some discussion of the latter, our main story is concerned with the two central triangles.

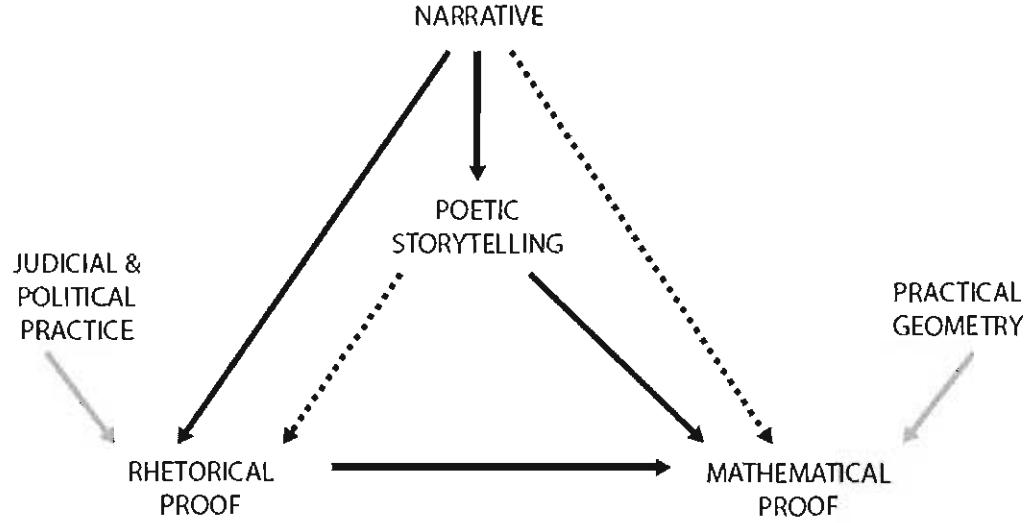

Figure 10.3. A more complete rendering of the relations shown in figure 10.2, taking into account the two different contexts in which proof develops.

2. Narrative and Poetic Storytelling

2.1. Narrative: Certain Cognitive Aspects

Though the terms *narrative* and *story* are often used interchangeably, it makes sense to distinguish between them, using the former to denote something more general; accordingly, all stories are narratives, but not all narratives are stories. Our capacity for narrative is a basic cognitive skill. Though this capacity is necessary for storytelling, it does not of itself explain the sophisticated form of many stories, which also requires a culturally developed practice.

In my use, "narrative" denotes a speech mode whose aim is *to represent action*.[4] According to philosopher and film theorist Noël Carroll, the basic constituents of a (verbal) narrative are sentences describing *actions* or *states-of-affairs* (Carroll 2001, 22). However, though a single action sentence (e.g., "John went to Paris") is a mini-narrative, a single state-of-affairs sentence (e.g., "The king is fat") is not, as it does not unfold in time, a condition necessary for action. Longer stretches of narrative can contain both kinds of atomic sentences, as state-of-affairs sentences enrich the representation of action. Another basic constituent of narrative, *dialogue*, can be subsumed into the two previous kinds, as it can be reduced to combinations of action and state-of-affairs sentences. We can add a third kind of constituent sentence to our definition of narrative, a thought in the form of *maxim* or, in the Greek context, *gnomê*. Though this can also contain a combination of action and state-of-affairs sentences, we distinguish it from those,

as it describes a general thought, ascribed in a narrative either to a character or a narrator.

Clearly, not all mixtures of the three kinds of constituent phrases form narratives. The basic advantage of our general definition of narrative, as representation of action, is that it does not need formal criteria to determine which combinations of phrases do and which do not constitute narrative. Instead, this is done by determining the ability of any concatenation of phrases to create a valid mental representation of action: if it has it, it is in the narrative mode; if it does not —though parts of it may be in the narrative mode—it is not. That there can be gradations of the criteria for narrativity from listener to listener does not annul this definition but grounds it in the human cognitive reality to which it should anyway belong.

In fact, it makes sense to speak of narrative as a symbolic mediation between two worlds, the *world of actions* and the *world of mental representations* (of actions). Experimental work conducted over the past two decades by cognitive scientists has brought forward the centrality of action in narrative (Johnson and Mandler (1980); Zacks and Tversky (2001)). Mandler and DeForest (1979) showed that when people hear a story they tend to model its recall on the underlying basic action, and not on the exact form in which it is told to them. A basic reason why narrative can best be defined with reference to represented action is that, of itself, "narrative is gappy; like a thin fabric. . . . If we start picking holes and looking at the gaps rather than at the smooth surface of narrative, then indeed it does start to fall to pieces" (Gainsford 2001, 3). In this, of course, narrative partakes of the nature of texts more generally, for all texts are gappy (Spolsky 1993).

It is highly relevant to any cognitively oriented study of narrative that "narratives are discourses that describe a set of actions" (Zacks, Tversky and Iyer 1999). Furthermore, the structure of the latter affects that of the former: "If actions can be thought about in terms of hierarchical part structures, it stands to reason that people apply these structures to understanding narratives" (Zacks, Tversky, and Iyer 1999) —and, let me add, to *creating them*.

2.2. Narratives and Narrative Worlds: Linearity and Nonlinearity

We can think of a basic narrative as a sequence of representations, strictly ordered in time.[5] The surface of narrative is thus *linear*, or *serial*. A stretch of narrative can be decomposed into narratively irreducible atomic phrases. Thus, "John ate a sandwich and then went home" is decomposed into "John ate a sandwich" and "then went home," the two events in serial order. These sentences are *atomic* in that they are temporally irreducible within the particular narrative: each one describes a unit of narrative action. Equally irreducible are state-of-affairs sentences, whose representations have no temporal duration (e.g., "The king is fat"), though they complement time-dependent action sentences in forming fuller representations, as in the second part of the narrative, "John ate the sandwich. It was ham and cheese."

Despite a narrative's surface linearity, however, the mental representations it encodes are often at least partly nonlinear. Even the simplest action that a narrative encodes in linear fashion is picked out of a nonlinear network. The cause of this nonlinearity is twofold: for each action, we can speak of its *outgoing nonlinearity*, which is determined by the many choices that can follow it. "The king died, and then queen died": even E. M. Forster's (1956) minimal example of what he calls a *story* is only the visible trace of a nexus of hidden, unrealized possibilities. Its surface form is clearly linear: it can be broken down in the two atomic sentences, as shown in figure 10.4, with the black arrow representing the passage of time. But this simplest of narratives is lodged in a huge world of possibility. The dotted lines in figure 10.5 indicate just a few other possible continuations to the first sentence, culled from what we know about the lives of kings. Thus, what makes even an atomic story cognitively interesting, in context, is that the events it describes *need not have been so*. In other words, a narrative becomes interesting in

relief, as it were, situated against its underlying world of unrealized possibilities.[6]

THE KING DIED. ⟶ THE QUEEN DIED.

Figure 10.4. A diagrammatic depiction of the temporal relationship of the two events in E.M. Forster's example.

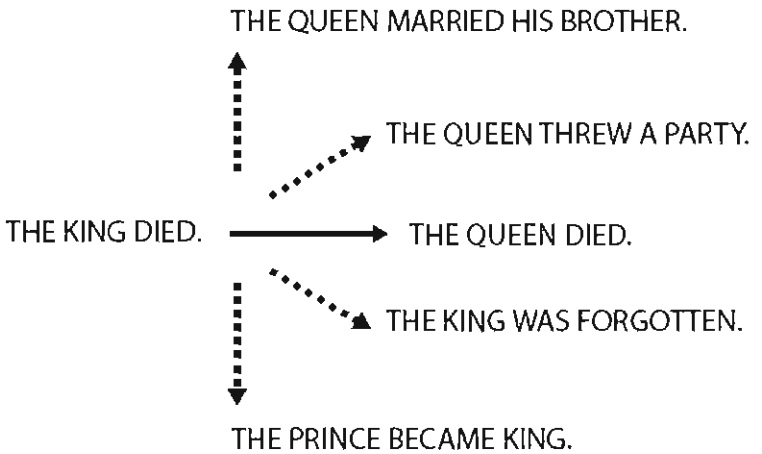

Figure 10.5. The two events of figure 10.4 shown inside a nexus of other possibilities.

Likewise, any event in a narrative has *incoming nonlinearity*. The reason for this is the complex nature of causality. Events in life are caused by chance and necessity; the occurrence of this action rather than that is determined by a combination of other actions and states of affairs: causality is multifactorial (Pearl 2000). Let's take E. M. Forster's next example: "The king died, and then the queen died of grief." The last two words give a (partial) cause for an action, which in a standard reading might imply hidden premise: "The queen loved the king." Even in this simplest of cases, however, the network of relationships cannot be linearly represented. For though the two events, "the king died" and "the queen died," can be temporally ordered, the reason for the queen's death—that is, the hidden premise—has no linear relation to the other elements. We see that neither of the two possible linear orderings works as explanation. The first fails because the king's dying was not the reason the queen loved the king (figure 10.6). But so does the second, since the queen's love did not cause the king's death (figure 10.7). To represent the causal relationships accurately, we need an extra dimension, giving the nonlinear causal diagram shown in figure 10.8.

THE KING DIED. ⟶ THE QUEEN LOVED THE KING. ⟶ THE QUEEN DIED.

Figure 10.6. In this attempt at a simple ordering, the middle element is incongruous.

THE QUEEN LOVED THE KING. ⟶ THE KING DIED. ⟶ THE QUEEN DIED.

Figure 10.7. The alternative ordering leads to a false conclusion (that the king died because the queen loved him).

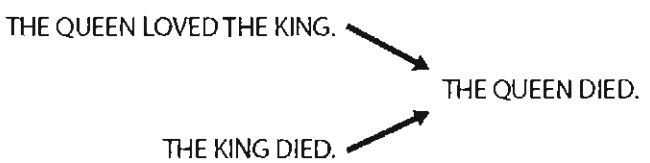

Figure 10.8. A non-linear arrangement of the three phrases gives a more accurate depiction of the underlying relationships.

There are additional reasons for incoming nonlinearity. The stereotypical explanation of the queen's love for the king as the reason for the queen's grief is not the only one that arises in a storyteller's mind: maybe the king and the queen lived in Rajasthan at the time the law of *sari* applied, whereby a widow would be burned alive on her husband's funeral pyre—and it can certainly cause terminal grief to know that this is your imminent fate; or the queen may have known that the king did not write a will before his death, thus leaving her at the mercy of a sadistic heir, and so on.

To recap: narratives are paths, linear sequences extracted from nonlinear worlds of relationships and possibilities. They are interesting because they delimit the range of the possible: at every instance, they pick one out of many. The "one" is linear; the "many," from which it is chosen, not so. For the hidden nonlinearity behind a narrative to show its face, we need either to look more closely at the causes of a particular event (incoming nonlinearity) or its possible alternatives (outgoing nonlinearity).

2.3. Dealing with Chaos: Goals, Modularity, and Scripts

Each narrative is culled from a much larger nonlinear world, which can be huge but is not necessarily limitless or totally formless. When a person acts, in life, fantasy, or narrative, he or she is guided by need, inclination, and available choice, and these already give some structure to the nonlinear, encasing world: if you are chased by a lion on a beach and (1) do not want to be eaten and (2) are without weapons, your options are to swim (if you know how) or take a boat (if there is one). But growing wings and flying or wrestling with the lion are not options unless you are a character in a particular type of non-realistic story.

The strategies for dealing with the complexity of nonlinear worlds come basically in three categories, *goals and subgoals*, *modularity and outlines*, and *patterns*. The first two are equally apparent in the world of narratives, while the third becomes much more important in stories, that is, narratives with a large amount of extra structure, basically culturally determined.

1. A *language of goals and subgoals*. The strongest complexity-defying characteristic of the world of action is its goal-orientation. Passions, needs, and habits act on the characters in narratives—as they do on us in life—to give action a forward thrust, thus limiting choice. Action is always at least partly goal-oriented, and thus structured in ways affected by the goal (Miller, Galanter, and Pribram 1960). A whole narrative toolbox can be develop around goals, regarding their achievement or nonachievement, their breaking down into subgoals, and so on (Doxiadis 2005).
2. *Modularity and outlines*. We can conceive of narratives as made up of parts, which can be telescoped into shorter versions. Though these outlines lose a lot of the information contained in the longer form, they are—partly for this very reason—helpful at a cognitive level, helping us to better understand the general shape of the action and thus to better orient ourselves in a narrative world at both the macro- and the microlevel.[7] The concept of

applied, whereby a widow would be burned alive on her husband's funeral pyre—and it can certainly cause terminal grief to know that this is your imminent fate; or the queen may have known that the king did not write a will before his death, thus leaving her at the mercy of a sadistic heir, and so on.

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3. *Patterns: scripts, formulas, motifs, scene type, story type, and genre.* As with any higher human mental activity, so with the human ability to compose and understand stories, the innate cognitive potential is enriched by cultural practice. As movement can develop into dance or dexterity into craftsmanship, so the underlying cognitive structures for language and narrative contribute to the development of storytelling. But storytelling is no more "natural" for human beings than singing or dancing. A "story-species" we may be (Gold 2002), but not in the sense in which seagulls are "flight species."

Pioneering cognitive psychologists Jean Piaget (Piaget and Inhelder 1973) and Ulric Neisser (1976) have proposed that action is governed by mental models, called *schemes*, that are structured and hierarchical. This research program and the notion of the algorithm were instrumental in forming the concept of *script* (Schank and Abelson 1977), or a generic narrative that both delimits and orders the action choices in a particular situation. The existence of scripts as guides for cognitive functioning indicates that a lot of the work of representation that goes into a narrative is already there before the action it describes has occurred: mental representations are not just mirrors held up to nature but molds for shaping it.

The needs and opportunities, beliefs, customs, mythical ideas, rituals, music, visual arts, and other expressions of any particular culture all contribute to the particular stories that it creates.[8] A traditional storyteller's craft is guided at all levels of composition by preexisting cultural patterns, often unconsciously employed. At the lowest level, verbal *formulas* are very frequent ("once upon a time"); higher up, we find standard *motifs* ("the three brothers"). Higher still, scenes are structured according to *scene types*, and types become even more important at the level of stories. In what is probably the most often-quoted study of narrative after Aristotle's *Poetics*, Vladimir Propp (1968) introduced the concept of *story type*, also identifying its variants for the subgenre of the Russian magical tale. A story type is a generic story with underlying rules and *constant* or *variant* elements. Guided by Propp's pioneering analysis, scholars in many related disciplines have sought, and found, similar type patterns in both traditional and literary narrative art forms. Finally, the familiar notion of *genre* gives us a typology of stories at the macrolevel.

2.4. The Greek Way: Poetic Storytelling

Though there are elements in folktale themes and style that indicate cultural universals (Thompson 1978; Lüthi 1982), we have no written evidence for prose storytelling of this type in Greece before the fifth century BCE. The first extant Greek artistic narratives are the epics of Homer and Hesiod (probably composed in the eighth and seventh century BCE, but written down later). The works of the lyric poets of the seventh and sixth centuries also have strong narrative elements. The dominance of the Homeric poems in Greek culture is an undeniable fact. The average educated Greek knew his Homer backward, and many stylistic features of the *Iliad* and the *Odyssey* played a crucial part in the formation of subsequent narrative forms, not only in verse but, later, in prose.

Preliterate verbal style is built on a syntax of nonsubordination (Goody and Watt (1963); De Vries (2005)). The style of most folk narratives usually entails *parataxis*, a form of composition in which independent clauses are concatenated, with or without a conjunction (Ong 1988). In archaic Greek literature we also often find *apposition*, with subsequent phrases augmenting the meaning of previous ones. Though the Greek language can contain periods of enormous syntactical complexity, these are built up from small sentences, which are themselves very simple (Perry 1937). John Dewar Denniston, writing of Greek prose style, states: "The edifice, lofty though it may be, is built of bricks, not of huge blocks of Cyclopean masonry" (1952, 61).

Building on our definition of narrative, we can say that the archaic storytelling style, especially as found in epic, works through a combination of innate cognitive mechanisms and the habits of a developing cultural practice, by putting together atomic narrative sentences, somewhat in the manner of a film editor combining shots to compose larger units of action, with the aim of creating a lively representation of action in the mind of the listener. This is not to deny the consideration in composition of aural, rhythmical, or musical effects or the importance of formal patterns. But it is important to stress that apart from small-or large-scale cultural patterning, which also works its own, medium-specific enchantment, the major guiding principle for the way simple bits of language are composed in stories is good representation.

Tragedy, the great new poetic form of the Classical Age, raises representation of action to a new level by resorting to full-blown *mimesis*. Attic tragedy, especially post-Aeschylean, is also in constant interaction with the budding prose art of rhetoric. There are rhetorical elements in tragedy, and tragedy influences the rhetoricians (McDonald 2006). The development of both genres is part of the larger story of the changes brought about in the life of the Greek poleis by political transformation, from tyranny to oligarchy to more participatory practices, the most advanced of which is democracy (Vernant and Vidal-Naquet 1988, 25). Central to this transformation are culturally developed forms of speech whose aim is persuasion. The story of the development of rhetoric in the fifth century BCE as a self-conscious practice involving trained professionals urges the central question: how does the new cultural need for persuasion affect the old methods of putting together atomic bits of language? The previous major criteria of composition were good representation and pattern-based aesthetic pleasure. How are these affected by the new needs of rhetoric? We cannot hope to answer this question without entering upon a more thorough discussion.

3. Rhetoric

There are three genres of rhetoric in the Classical Age, each one arising from different needs, and thus each also creating its own solutions. These genres are the *epideictic*, or speeches usually delivered on festive occasions or at oratorical exhibitions; the *deliberative* or *political*, used by speakers in the assembly; and the *forensic* or *judicial*, spoken by litigants in a court of law. Distinguishing the formal characteristics of the three kinds, Aristotle wrote, "Amplification is most proper to epideictic . . . past fact to judicial . . . and possibility and future fact to deliberative speeches" (*Rhetoric* 2.18.5). Both "past fact" and "possibility and future fact" are intimately related to the narrative mode.

3.1. Epideictic and Deliberative Rhetoric: Influences from Narrative and Poetic Storytelling

Of the three genres, I shall focus mostly on forensic rhetoric, as it offers the clearest examples of the type of thinking that is also relevant to mathematics.[9] But first I will discuss some interesting aspects that are seen more clearly in the other two genres. The first is from epideictic rhetoric, which is a direct development of lyric poetry.

3.1.1. Rhetoric Combines Narrative with General Principles

It is public-occasion rhetoric, meant in theory either to praise or to blame but almost always employed, in practice, for the former purpose.

The passion for proverb-like *gnômai*, or general opinions, is an essential aspect of the ascent

of the new, rationalist mentality in classical Athens (Johansen 1959). Though only representing a little over one percent of text in the *Iliad*[10] (Lardinois 1997), general opinions become increasingly prevalent in lyric poetry and even more so in tragedy (Martin 2005), to rise to a peak of nearly fifteen percent of the text of the tragedies in Euripides' middle period (de Romilly 1990). Though *gnômai* in earlier poets are culled from common lore, in the latter part of this era many are invented by the poet using them. These constructs follow the outer form and function of *gnômai*, but are essentially new creations. (Lang1984) stresses the difference in the kind of *gnômai* used in the works of the more traditional Herodotus, who worked with old and accepted forms *gnômai*, and the "more modern"—that is, also more sophistical—Thucydides, who mostly invented his own *gnômai*-like statements.

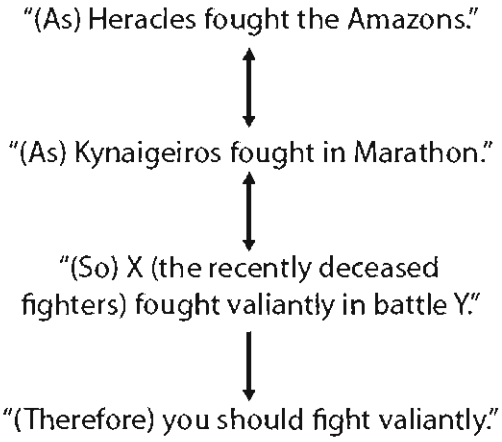

Figure 10.9. Moving from a sequence of similar exemplary stories (*paradeigmata*) directly to an injuction.

In Homer and lyric poetry, general principles are usually injected into the poem to stress the analogy with prototypical stories. This is done either through mythological *paradeigma*, in Homer (Willcock 1964), or in the form known as *comparatio paratatica*, in both epic and lyric poetry (Johansen 1959). In these uses the older story functions as guidance, in the form of "as so-and-so did, let us also do." Extensive use of this pattern is made in the lyric poems of praise and carried over to the epideictic form of funeral oration. In this, usually not one but a series of *paradeigmata* of similar content is given, from which the rhetor passes to an injunction. For example, exhorting the youth of the city to emulate the heroic dead fallen at a certain battle, one might say the words shown in figure 10.9. As the Classical Age progresses, however, an extra link is often added in this abstract scheme, in the form of a *gnômê* distilling the *paradeigmata* into a principle or rule (figure 10.10). However, in the most famous extant funeral oration, Pericles' *Epitaphios* as transcribed by Thucydides, narrative *paradeigma* is dropped altogether. Pericles declares that he will dispense with the usual narrative accounts of old military exploits, as they are long and familiar to the audience. Instead, his speech is filled with *gnômai*-like sayings. These lead directly to injunctions, as for example when the gnomic "judging freedom to be happiness and courage to be freedom" is followed by the admonition to the young not to "be too anxious about the dangers of war" (*History* 2.43.4). Such a simplified sequence is shown in figure 10.11. This speech might lead us to the conclusion that Athenian democratic practice, at its peak, solely deals with abstract principles. But we only have to see Pericles'-Thucydides' *Epitaphios* in the context of earlier epideictic rhetoric—as the exception, rather than the rule—to understand that this abstract form is a development of narrative thinking.

Figure 10.10. Unlike the sequence in figure 10.10, the sequence of *paradeigmata* leads to a general rule; the injunction then follows as its application.

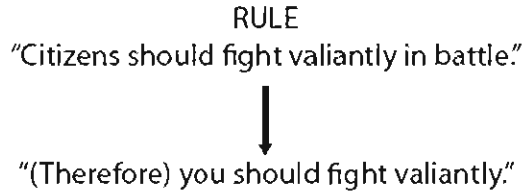

Figure 10.11. The *paradeigmata* are dropped altogether, as the speaker moves from the general maxim to the injunction.

3.1.2. Narrative as the Representation of Action

Unlike epideictic rhetoric, which is to a large degree a transformation of a poetic genre into prose, the use of oratory in politics—as also in courts of law—develops new forms of its own. In these, some of the basic cognitive qualities of narrative are realized to their full potential. More specifically, in deliberative rhetoric we see narrative as the representation of an action.

In the *Institutio Oratoriae*, Quintilian, a Latin theorist building on Greek precedent, writes, "Narration is the exposition of that which has happened, or is supposed to have happened, *with the intent to persuade*" (4.2.31; my translation and italics). Nowhere is this clearer than in Thucydides. The aesthetic dimension of narrative is diminished in this new art of narration. Of course, a historian will apply verbal craft, to make an outwardly unadorned style more convincing by the use of this or that word, an adverb less or more. But historians' narratives are a world apart from poetic ones, whether epic, lyric, or—less so—dramatic, in which the legacy of verse, as well as a mythological or heroic subject, guides the poet, who occasionally works more on the icing than on the cake itself.

Not so Thucydides. In the *History*, completed between 410 and 400 BCE, the great political decisions facing the polis, such as the formation or breaking down of an alliance or a new military campaign, are reached through battles of speeches. In these the opposing political sides present their narratives of possible futures and attack their opponents'. These verbal battles exploit to the full the mechanisms of *narrative intelligence* and what cognitive scientiests call *theories of mind*, in other words, the skills required to understand stories as representations of action and analyze them in terms of motives and causes; or, in our terminology, outgoing nonlinearity (possible outcomes) and incoming nonlinearity (causes). At the heart of these

cognitive abilities is the notion of *narrative probability*, a form of knowledge, honed by experience, of what is more or less possible in reality—an art that is statistical, not absolute. A general or statesman trying to sell his views on a battle plan knows that his narrative of how the battle will develop will carry a big part of the argument. Thus, the more plausible it sounds to the ears of his listeners, the more likely it is that it will offer the most convincing scenario of future events.

3.1.3. Convincing Narratives Are Heavy on Causality

The art of Thucydides' narratives, which "with their stark power and exactitude, remind us of a theorem" (de Romilly 1967; my translation), is the art of making one's narratives of the future seem necessary. This is achieved by showing that their premises are correct, and the ensuing calculus of causality is realistically worked out. Central to this is the paring down of the narrative to only those events that are directly involved in causal chains, as this tends to create a greater impression of "necessity". The conceptual context of these narrations are shared assumptions, both general and specific—the former are about human nature more generally, while the latter about the present condition of the polis—as well as knowledge of the enemy's words and actions, interpreted through theory of mind, in the cognitive science sense. These form the web of incoming and outgoing nonlinearity inside which a future narrative, or scenario, can be best defended as optimal.

3.1.4. Narrative Is Goal-Oriented—Because Action Is

Of course, actions have unpremeditated consequences, and unknown factors play a big part in human affairs: neither incoming nor outgoing nonlinearity is totally fathomable. Thucydides was fully aware of that, and thus his view was essentially tragic (Cornford 1907). However, despite this—or perhaps because of it—he rarely abandoned the effort to depict his human actors as involved in a process of rational discussion, thus generating de Romilly's "theorem-like" narratives.

Goal orientation of human actors is a necessity in future narratives—as it is in fiction, for that matter. When recounting the past, the narrator is constrained by the actual. Though he or she may add, subtract, or reinterpret this or that fact, the larger set of events from which the narrative is culled is given: things have happened, mainly, because they have happened. Not so with a future narrative. A strong sense of goal is necessary for a speaker not to lose his way in the mental labyrinth of the outgoing nonlinearity of all possible outcomes. It is in this sense, too, that Thucydides' narratives have the taste of theorems: like them, they trace well-determined paths from what is known, the existing situation, to what is to be "proved": the particular version of future event that the speaker thinks the city ought to be aiming at. The difficulties in proving theorems lie in finding a way to get from the premises to the conclusion. It is the same with future narratives, and the more causally necessary this process seems, the more convincing it will be to the listeners. In fact, if we replace "causal" links in Thucydides' narrative arguments, with "logical" links—as, for example, in changing "B was caused by A", to "B follows from A"—we can see that the structures of his historical thinking are often exact templates of those of the deductive arguments of a logician or mathematician.

3.1.5. Narrative Is Modular

The property of modularity is particularly useful when dealing with the future: when constructing plans of action, it is good to know that certain areas of possibility are independent of others, that is, that it is *not* true that every event can influence every other event—otherwise the number of possibilities would be enormous and there would be no way to rationally deal

with any of them. This is a fact that Classical rhetors are well aware of, and often resort to, in structuring their speeches.

To recap: though treatment of narrative in epideictic rhetoric is not far removed from the poetic, in deliberative rhetoric it returns to a function closer to its cognitive grounding in the representation of action. Many techniques of poetic storytelling are abandoned in favor of the basic cognitive qualities of narrative. The fact that a political speech occurs in an agonistic setting is crucial to this: the cultural pressure that makes a political orator rely on the essential cognitive qualities of narrative is the fact that the other side will contest it. It is through these essential cognitive qualities that a rhetorical narrative's "intent to persuade" (Quintilian) is realized. By its very structure, and principally by its *economy in the service of its central causal claims*, a good rhetorical narrative is halfway to being a logical argument, the logical connections here appearing as causal relations.

3.2. Forensic Rhetoric: The Problematics of Stories

Rhetoric had its origin to a considerable extent in the attempt to give to prose the same qualities of beauty which its elder sister, poetry, already possessed.

—Samuel E. Bassett, "Hysteron Proteron Homerikos"

The cognitive qualities of narrative that shape deliberative rhetoric are also crucial in the development of its forensic sibling. On the surface, narrative in forensic rhetoric seems to be devoid of the qualities of poetic storytelling, mostly relying for its effectiveness on convincing representation of action. But this is only partly correct. Viewed differently, and especially when examined at the level of microstructure, forensic rhetoric is centrally about the transposition into prose of many of the tricks of poetry. The surprising thing is—and this is one of my main contentions—that these poetic qualities are almost absent from the narration at the heart of the rhetorical speech and much more apparent in the part of it we are accustomed to viewing as most "logical," the part containing the direct argumentation.

A sort of paradox is involved here, a paradox that may contain the heart of the early history of logic: though the aim of the orators importing poetic techniques into rhetoric could well have been originally aesthetic, as stated in Bassett's astute observation heading this section, those same techniques soon became the basis of the budding method of logical argumentation. We could describe this change with a term from evolutionary biology as *exaptation*, or the use of a feature originally "evolved for other usages . . . and later 'co-opted' for [its] current role" (Gould and Vrba 1982, 6). This concept is central to my argument: what I am attempting in this essay, and more particularly in this section and the next, is a history of the emergence of logic as (1) making the most of the basic cognitive qualities of narrative, within an agonistic setting, and (2) developing a series of exaptations of poetic techniques in the service of persuasion.

More precisely in the context of forensic rhetoric, my thesis is that what later became known as logic began to develop as a method for comparing contesting narrative accounts of events, with the aim of persuading a jury of citizens that the speaker's version provided the best fit to the facts. It is crucial for this argument—especially in view of its extension to the emergence of mathematical proof—to remember that rhetoric hardly ever works with all-or-nothing statements. It is precisely this that causes a part of it *not* to overlap with the logical methodology of mathematics (see figure 10.1). The truth-values of rhetorical statements are not restricted to zero (false) and one (true) but move also in the range in between, according to the central Greek concept of *eikos*.[11] These intermediate truth-values depend on *narrative probability*, which becomes crucial in situations where either only some of—and not all—the facts of a case are known or where certain of the facts are given in more than one, contrasting versions. In other words, narrative probability is crucial in most realistic situations.

The narration of the facts of a case plays an important part in a forensic speech to this day. Perhaps because of the deceptive "naturalness" of narrative, though, ancient theoreticians—with the exception of Quintilian[12]—did not pay it the attention it deserves as a tool for persuasion. It is only in the past three decades that legal scholars have begun to examine closely the role of narrative in legal thinking and practice,[13] though, as Gagarin (2003) points out, these studies have not, with only the rarest of exceptions, been extended to the study of ancient law.

3.2.1. The Context of Greek Forensic Rhetoric

As previously mentioned, tragedy and deliberative rhetoric are best examined within the new structures of political organization appearing in the Greek world in the late Archaic and early Classical Age, structures which gave a high value to dialogue, rational deliberation, and collective decision making. The same applies to the examination of forensic rhetoric.

In a classical-age trial, the outcome was determined by a jury, consisting of some hundreds of free citizens who voted secretly after listening to witnesses and speeches from the two sides. The prosecution in public cases was usually conducted by a volunteer (*boulomenos*) while the defense was conducted personally by the defendant, though the latter's, and sometime even the former's, speeches were written by a professional orator called, in this case, *logographos*. Though in some cases each side was allowed one speech, in homicide cases and many other cases there were four, one speech and one rebuttal for each side (Carawan 1993).

The raison d' être of a trial is a series of events that actually occurred. A forensic speech is basically constructed with the aim of persuading a jury that the speaker's narrative of these events provides a better fit to reality than the other litigant's. The new, Classical mindset, which forms the proper context for the speeches we shall consider, includes the realization that there does not exist a one-to-one correspondence between narratives and events, and that the same events can generate various, and often conflicting, narratives.[14]

The earliest signs of this mentality are to be found in the work of those lyric poets who begin to play around with old myths;[15] the fifth-century tragic poets absolutely reveled in such revisionism, both creating new variations of the old myths and reading new meanings in the old. These practices are marks of the awareness that a narrative is not *the* representation of certain events in a storyworld but *a* representation. The agonistic environment of tragedy and forensic rhetoric—the former possibly inspired by the latter—leads to the further realization that figuring out which of the alternative narratives for the same set of past events gives a better fit to reality is nonobvious and problematic. Figuring out what actually happened is not just difficult for the poet (in tragedy) or the rhetor (in a court of law) but also, respectively, for the audience or jury: an essential part of the art of citizenship of a polis is constantly facing up to, and deciding between, such difficult choices between conflicting narratives.

3.2.2. The Macrostructure of Persuasion: Parts of the Speech

The role of narrative and poetic storytelling in the process of demonstration in Greek forensic rhetoric can be examined at the levels of macrostructure and microstructure. For the first, we shall look at the generic structure of the forensic speech as it developed from the second half of the fifth century BCE to its high point, less than a century later, and especially as it was theorized in Aristotle's *Rhetoric*.[16] The flexible generic structure underlying forensic speeches is already forming in the earliest extant texts of speeches by Antiphon and Gorgias (ca. 430–420 BCE) and is more or less fixed in the late work of Lysias and Andocides, a couple of decades later. It is important to begin by stating that the very existence of a generic template for the speech is in keeping with the tradition of poetry: almost as soon as a new literary subgenre is born, it acquires its own particular standard form.

The template of a forensic speech has four parts, the *introduction*, the *narration*, the *proof*, and the *epilogue*, although some ancient theorists further break down the third part into *division*, the *proof proper*, and *refutation* (de Brauw 2006, 187). The aim of forensic speech is twofold: to create relative, rather than total, certainty that the rhetor's narration best fits the facts of the case; and to show that the acts described in it are best to be judged by application of a particular law of the polis.

3.2.2.a. Introduction (*prologos*)

Apart from the aim of capturing the audience's attention and impressing it with a demonstration of the speaker's verbal ability, an important function of the introduction, especially as theorized in Aristotle's *Rhetoric*, is to define what is to be proved, in terms of both narration and applicable law. For this, it puts forth an outline, which can be as short as one sentence, of the speaker's version of the story and states his contention as to the applicable law. This function of an introduction is not invented *ex nihilo*: outlining a story's plot inside the story is an old, tried strategy of poetic storytelling, probably developed as an antidote to the sprawling, digressive epic style (Notopoulos 1951). Though it can be argued that outlining within the story is a case of repetition, and as such a basic mechanism in the unplanned "poetics of talk" (Tannen 1987), more particularly the starting of a narrative performance with an outline of its plot is a standard epic trick. In fact, the similarity of this function of the forensic speech to the similar poetic practice is noted by Aristotle (*Rhetoric* 1415a). Famously, both Homeric epics begin with a brief programmatic outline of their story and its meaning. When it precedes a sequence of events, the outline of a part or the whole of the plot in an epic fulfills the additional cognitive task of giving the narration forward thrust, order, and unity, through anticipating or recapitulating the action (van Groningen 1935; Notopoulos 1951).[17]

Not all the introductions in extant fifth- and fourth-century speeches conform to Aristotle's generic pattern of setting out the main argument. But those that do could well be emulating epic practice: the employment of a preexisting, handy trick, used to cover a somewhat similar, but essentially different need in a new context—an exaptation.

3.2.2.b. Narration (*prothesis* or *diêgêsis*)

This part of the speech is pure narrative, a seemingly straightforward setting out of the facts of the case being judged. The art of the speaker lies in doing this in the way that best serves his purpose: in selecting from the larger storyworld of the case the appropriate facts and combining them in a way that highlights the causal relations crucial to his own interpretation of them.

The fourth-century rhetorician Alcidamas states in *On the Sophists*: "Speeches which have been laboriously worked out with elaborate diction, compositions more akin to poetry than prose [*logoi*], are deficient in spontaneity and truth, and, since they give the impression of a mechanical artificiality and labored insincerity, they inspire an audience with distrust and ill-will" (15.12.1; transl. Van Hook 1919). This is particularly true of the narration. But the fact that the artistry in a narration should not proclaim its presence does not mean that it should not be there. On the contrary, as human beings tend to read the flux of everyday reality in terms of preexisting story patterns, the skilled rhetor may cunningly (which also means *invisibly*) impose such a pattern on a case's events in order to augment their narrative probability. He may even, consciously or unconsciously, employ an existing narrative pattern; thus, Porter (1997) suggests that the narration in a murder case by Lysias follows patterns of fourth-century theatrical comic storytelling. Clearly, the imposition of a preexisting pattern on a legal narration is not always in the service of the truth—though it is always in the service of effective persuasion. In fact, it can have a desultory effect on it: as reality is as a rule a lot more unstructured, random, incomplete, and inchoate than any well-formed story mold, the imposition of an external narrative pattern

on the facts of a case often works to distort the truth rather than illuminate it (Dershowitz 1996).

In an unjustly neglected book, O'Banion (1992) makes a strong case for the importance of *narration-as-proof* in ancient legal procedure, often quoting Quintilian (ca. 35–ca. 95), the ancient theoretician who above all others made this point strongly. Quintilian's definition of narration, quoted earlier, points to the fact that the change of emphasis in Classical Greek narrative techniques, from purely poetic storytelling to the great prose narratives of the orators and the historians, may lie in the intention with which a narrative is set out: not to delight, entertain, or instruct but to convince. Thus, the degree of detail in the narration of a forensic speech and the choice of parts of the story to be told or suppressed are determined by what they are there for—in other words, persuasion. Form follows function.

In the strongest statement of the narration-as-proof view, Quintilian writes: "For what difference is there between the proof and the narration, other than that the narration is a proof in continuous form, while the proof is a verification of the facts in the narration?" (*De Oratore* 4.2.79; my translation). Thus, authors adapting ancient terminology to modern legal usage, and translating the Greek *diêgêsis* and the Latin *narratio* as "statement of the facts", seriously distort the spirit of the original terms, which reflect more strongly than any theoretical statement the affinity of ancient legal thinking—and thus early logical thinking—to its narrative roots.

Following the demand of simplicity-as-guarantee-of-veracity, the *diêgêsis* of a forensic speech is on its surface as simple as possible. Any hidden artistry notwithstanding, it is a stark, "documentary" recording of experience,[18] containing no direct dialogue. Narrative of this kind joins together basic atomic sentences of two types, action and states-of-affairs, to form larger blocks of action or scenes.[19] Atomic sentences are put together through links that either define *temporal* or *spatial* relation (such as "and then" or "elsewhere"), *causal* ("and so"), or *associative* of various kinds ("which brings to mind," "who was the son of," etc.) As mentioned in section 2.2.2, the early poet-storytellers structured their narratives almost exclusively through a syntax of nonsubordination, either parataxis or apposition, in a "strung-along style" that affects the "very mentality of the poet" (Notopoulos 1951, 87).

As a forensic narration does not aim at objectivity, the speaker seeks to represent the actions being judged in the form that best suits his purpose, while at the same time respecting the principle of high narrative probability and thus believability. In fact, believability is the one quality ancient theoreticians agree a forensic narration ought to have: it must give a representation of action that convinces it could be true or, more precisely, that cannot be convincingly shown to be untrue.

3.2.2.c. Proof (*pistis*)

A somewhat Whig-historical view of Greek intellectual life as striving toward the condition of logic makes the *pistis* the most important part of the speech. But it is not. The core of the speech is the narration; the *pistis* is only there to support it.

As mentioned earlier, some ancient theoreticians[20] further decompose *pistis* into three parts: *division*, the actual *pistis*, and *refutation*. However, regardless of whether we adopt this further breakdown or not, the proof (*pistis*) of every particular argument is based on the rationale of division, and thus it makes sense to examine this process separately. (On the other hand, refutation of the opponent's claims, either actual or potential, is the motivation behind the structuring of the whole *pistis* part of a speech—if a part of the narration is not to be contested, it isn't worth spending any time trying to prove—and thus we don't need to discuss it separately.[21])

3.2.2.d. Division (*diairesis*)

The first task of division is to break down the action in the speech's narration into parts,

selecting those, or the interpretations thereof (of an action's motive, say), which actually need to be proved, or—what is really the same thing—which the other side may contest. The second is to put forth for every contestable part of the action—which always has the form of a small narrative—a list of alternative narratives that the opponent may propose as likelier, to contest it. By this twofold process, the division brings into the speech *new narratives* that the speaker will use, in order to establish beyond reasonable doubt that his own narration gives the best approximation of the truth.

There are two different kinds of narratives that division generates in this process: *subnarratives* and *counternarratives* of the main narration, both of which are created by building on the basic cognitive mechanisms of narrative intelligence.

3.2.2.d.1. *Subnarratives.* To create these, the speaker deals with the main narration piecemeal, examining certain of its parts in greater detail to expose possibilities lying beneath the surface. The speaker muses on the causation of the various actions described, which is often tantamount to addressing the actors' motives. The driving idea behind this process is the realization that, in order to be effectively contested, or even intelligently discussed, a narrative may need a certain extra amount of detail. A good example of this process is given by the division in the *Encomium of Helen* (ca. 430 BCE), Gorgias's display piece exonerating Helen of her alleged responsibility in starting the Trojan War. The speaker addresses the central action, Helen's departure from Sparta with Paris, which is usually given in the legendary accounts without an explanatory motive. Gorgias, however, does not, like Stesichorus in the *Palinôdia*, go for an "alternative history" explanation, involving abductions by gods and such. He accepts the basic action of the legend as fact and brings more detail into its representation by asking what caused Helen's action. The possible scenarios offered are consistent with the folk cosmology and folk anthropology of a citizen of classical Athens, in other words, what is narratively probable in the mind of his audience.

This is the encomium's brief division (my translation and numbering; Diels-Kranz 289):

> Either by [1] the wishes of fate and designs of the gods and decrees of necessity she did it or [2] she was taken by force, or [3] persuaded by speeches, or [4] conquered by Eros.

Gorgias does not put forth one of the four options as more probable than the others: what is important in his attack is that the four, taken together, exhaust all possibilities. If the orator then manages to deal with each one of them successfully in the proof proper, he will have effectively dealt with the whole problem.

3.2.2.d.2. *Counternarratives.* The second type of new narrative that division can create is the *counternarrative* to a part of the main narration. Stesichorus's story of Helen's divine abduction is a good example, though the speaker in a classical courtroom would opt for counternarratives with a higher degree of narrative probability. In the following example of the creation of counternarratives, the division is not set out in a separate section of the speech but spread over the *pistis*. It is from Antiphon's *First Tetralogy* (ca. 430 BCE), a textbook criminal speech. The defendant is accused of having killed a man in a secluded spot, with no witnesses except the dead man's slave, who named the defendant as perpetrator before dying of wounds incurred during the incident. The defendant was at that time in serious litigation with the victim, which he was very likely to lose, along with his fortune. This points to a strong motive for the crime. The prosecution's thesis in this, its first speech, is that the defendant must have killed the victim, as all other alternatives are less likely. To prove this, the prosecutor lists possible alternatives to the defendant having performed the deed, attacking each one as he goes along, ending in a sort of reductio ad absurdum. The parts in (my) italics set out the four counternarratives (115.4; transl. K. J. Maidment):

> *Malefactors are not likely to have murdered him*, as nobody who was exposing his life to

> a very grave risk would forgo the prize when it was securely within his grasp; and the victims were found still wearing their cloaks. *Nor again did anyone in liquor kill him*; the murderer's identity would be known to his boon companions. *Nor again was his death the result of a quarrel*; they would not have been quarrelling at the dead of night or in a deserted spot. *Nor did the criminal strike the dead man when intending to strike someone else*; he would not in that case have killed master and slave together. As all grounds for suspecting that the crime was unpremeditated are removed, it is clear from the circumstances of death themselves that the victim was deliberately murdered.

I propose that the type of thinking that drives the counternarrative part of the division has an earlier model in the common archaic poetic form of the *priamel*. This form, whose first extant examples are in Homer, is "a series of statements (often examples) [that] are advanced only to be ultimately set aside in favor of a culminating statement (the superlative example)" (Johnson 2008, 231). Here is an example from Sappho (L-P fr. 16; my translation):

> Some say that cavalry, some infantry,
> some a fleet, is the most beautiful [thing] on the dark earth.
> But I say [it is] one's lover.

This is a typical priamel in its structure: the first part offers the so-called *foils*—cavalry, infantry, fleet—while the second gives the *winner*, one's lover. This structure of foils-followed-by-winner is exactly parallel to the extract from Antiphon's speech: the rhetor produces a series of counternarratives (foils), only to discard them in favor of the event in his own narration (winner). This is a strong candidate for a cultural exaptation, from poetic to rhetorical use: a device originally used for emphasis, now used for persuasion.[22]

3.2.2.e. The Proof (*pistis*) Proper

The proof section of a classical forensic speech uses two techniques for producing conviction, the so-called *nonartistic (atechnoi)* and the *artistic (entechnoi) pisteis* (McDonald 2006, 195).[23] In Greek theoretical treatises, *pistis* as a rule denotes both the whole proof part of the speech and its parts; thus, one can refer to "the" *pistis*, denoting the whole of the proof part, but also "a" *pistis*, i.e. a particular argument, inside it. When denoting the latter, I follow the practice of Kennedy (2007, 31) and Gagarin (1990, 24) in using the word *pistis* in its original form, rather than translating it as "proof," as some authors do; I use "proof" only to refer to the part of the speech containing the individual *pisteis*.[24]

Nonartistic pisteis include witnesses' statements, given either under oath (free men) or torture (slaves), as well as contracts and laws; the nonartistic *pisteis* contain information that purportedly comes to the speaker from outside, without him adding any intellectual value to it, as it were.[25] The information in witnesses' statements is either directly or indirectly in the narrative mode; for even laws and contracts are also as a rule expressed in action- and state-of-affairs statements, in the form of either positive or negative injunctions, as also in general rules referring to them. Thus, rather than a purely narrative sentence of a witness, such as "*Eratosthenes was having an affair with the defendant's wife*," in a law we get a hypothetical narrative, such as "*if a man is having an affair with a married woman. . . .*"

The civic and political reforms of the late sixth and fifth centuries BCE, which awarded equal credence to the testimonies of all free male citizens, regardless of social status, or rank, created the need for the court to be able to compare testimonies in order to choose between them. This gave added value to the artistic *pisteis*.[26] Unlike the nonartistic *pisteis*, which are purportedly imported into the speech from an external reality, artistic *pisteis* are expected to be invented by the speaker by Aristotle's scheme. These come in three varieties: *pisteis* relating to the character

(*êthos*) of the speaker, opponent, or witness, *pisteis* addressed to the jurors' emotions (*pathos*), and finally the *logos pisteis*, which are addressed to reason (*dianoia*). Today, we are used to regarding these latter *pisteis* as "logical proofs." However, the original meaning of logos-type *pistis* is closer to the fact that these "lie in the speech itself, either demonstrating [*deiknunai*] or seeming to demonstrate" (*Rhetoric* 1356a2–4; my translation). The word translated as "speech" here is *logos* and this is significant: the original criterion for what we call "logical" *pisteis* today, thus all but equating them with rational thought, could be originally simply understood to mean "purely based on the speaker's words." (But with a caveat: this restriction to the speaker's words does not exclude the wider, nonlinear storyworld containing the action represented in the trial's narration or the additional knowledge that an average citizen could accept as naturally resulting from it by applying the standard mechanisms of natural narrative intelligence.)

3.2.2.f. Epilogue (*epilogos*)

The meaning of the Latin root *perorare*, which provides an alternative English name for the epilogue (*peroration*), is "to speak at length," a good indication that a lot in an epilogue is redundant, in terms of either information or argument. According to the anonymous *Rhetorica ad Herennium* (ca. 80 CE), an epilogue can be further broken down into a part outlining the case, or *summing up*, an *amplification* of the main argument, as well as an appeal to pity or *commiseration*. The last two are basically pathos-type arguments and aim at stressing to the jury the significance of making the right decision, usually supported by one or more *gnômai*; the function of these two parts can be seen as closer to poetry than any sort of logic. The summing up, on the other hand, basically repeats the basic narrative premise at the heart of the speech, as also set out in the introduction.

4. The Microstructure of Persuasive Discourse

Apart from the *diêgêsis* itself, which is pure narrative, the parts of the speech that are mostly narrative are the prologue, the division part of the proof, and the summing up of the epilogue, the first and last in the sense of outlining the action and the division in using narrative intelligence to generate from the narration new sub- and counternarratives. This leaves one important part of the speech to be examined for its possible relations to narrative and poetic storytelling, the *proof proper*.

As we saw, the proof proper operates piecemeal, dealing in turn with each one of the sub- or counternarratives defined by the division, with the aim of either making those supporting the speaker's narration appear more plausible or those contesting it more implausible. To understand how this is done, we must turn to the microstructure, and especially to the logos-type *pisteis*. It is in these that we find the roots of the process that became, over a period of some decades, what we today recognize as "logic". Bassett's earlier observation gives us a strong key to the genealogy of these artificial—in the sense of not naturally resulting from narrative intelligence —tools. I propose that the artistic *pisteis* enter fifth-century rhetoric from earlier uses in poetry and poetic storytelling. They were probably initially imported to relieve the dryness of prose with heightened linguistic constructions. However, some of these forms proved more useful than others in the main function of the *pisteis*, i.e. persuasion, and thus become increasingly dominant. In other words, rhetoric borrowed the forms of poetry for their beauty but kept, and further developed, some of them for their persuasive effectiveness. I argue more specifically that to construct their arguments, and especially the logos-type artistic *pisteis*, Greek orators made extensive use of two related techniques from archaic poetic storytelling, *chiasmus* and *ring-composition*. The atomic logical structures later theorized by Aristotle as syllogisms are, in fact, versions of these two techniques of poetic composition in their basic form. But the two techniques are also responsible for the architecture of logos-type

constructions at higher levels of organization of the proof part of a speech as will be shown.

4.1. Chiasmus and Ring-Composition: From Old Form to New Function

Chiasmus and ring-composition provide the missing link, the most crucial ingredient in understanding the passage from speech aimed solely at narrative representation to (what was later called) logical deduction. However, to avoid any teleologic-sounding arguments, I will avoid as much as possible the use of the word *logic* and its derivatives in what follows. Rather, to understand the importance of this transformation, I return to the streetcar metaphor: proof is not a sudden break with tradition but—in a process that is extremely common in the history of ideas, science, and technology—the putting to new uses of old tools in order to address a new need, solve a new problem.[27] The new need in this case is the employment of speech (*logos*) not merely to represent action but to compare narratives, with other narratives and to check whether they conform to general rules, the narrative- or *gnômê*-like laws of the polis.

Some scholars occasionally use the terms *chiasmus* and *ring-composition* interchangeably, and this is partly justified as they are nearly the same, but for one difference: the existence in a ring-composition of a unique central element; others reserve chiasmus for very short forms only. Despite their great similarity, however, it is essential to distinguish between chiasmus, which I abbreviate as X, and ring-composition, which I denote as RC; when an argument I make applies to both, I shall write X/RC.

X is a symmetric structure of phrases that, in its minimal form, has the structure A-B-B*-A*, where B* and A* repeat the whole or part of B and A, respectively. Any N is connected with its N* through a common ("white" joins up with "white") or antithetical ("white" joins up with "black") element, which can be a word, phrase, or concept. We shall call this common element the pair's *pivot*. A famous modern example of the minimal X is John F. Kennedy's "ask not what your country can do for you; ask what you can do for your country." Here, "country" is the outer or A-pivot and "you" the inner or B-pivot. An X can be of arbitrary length: A_1-A_2-A_3 . . . A_N-A^*_N- . . . A^*_3-A^*_2-A^*_1. RC differs from X solely in that the central pair N-N* is replaced by a unique central element N, or, in the simplest possible form, A-B-A*. For simple examples of rather short RCs we can look at those limericks where the first line is repeated at the end; here, these provide A and A*, while the central three lines are B. An RC can also be of arbitrary length: A_1-A_2-A_3 . . . A_N- . . . A^*_3-A^*_2-A^*_1. X/RCs can also be many-tiered, as in the RC where an element A_i further breaks down into A_{i1}-A_{i2}-A_{i3}... -A_{iN-1}-A_{iN}-A^*_{iN-1} . . . A^*_{i3}-A^*_{i2}-A^*_{i1}. In such tiering, X and RC can combine: an X can contain elements that break up into RC, and vice versa.

The first exact instances of X/RC in Greek literature are in the Homeric epics. These were first systematically defined and discussed by van Otterlo (1944) and further studied by Van Groningen (1958). In epic, X/RC operates at many levels of structure of the poem, from the smallest to the largest, as also at many levels in between, such as that of type scenes.[28] A strong reason for the use of X/RC in epic is its mnemonic value (Rubin 1996). As mnemonic techniques are often structured on an internally represented spatial form, the existence of a going-there-and-back spatial model could be at the root of X/RC. It is worth noting that recent evidence from the neurosciences points to the grounding of the spatial art of memory, the *technê mnêmonikê*, on the structure of the brain (Becchetti 2010, 104). This could also provide the basis for understanding the cognitive power of X/RC.

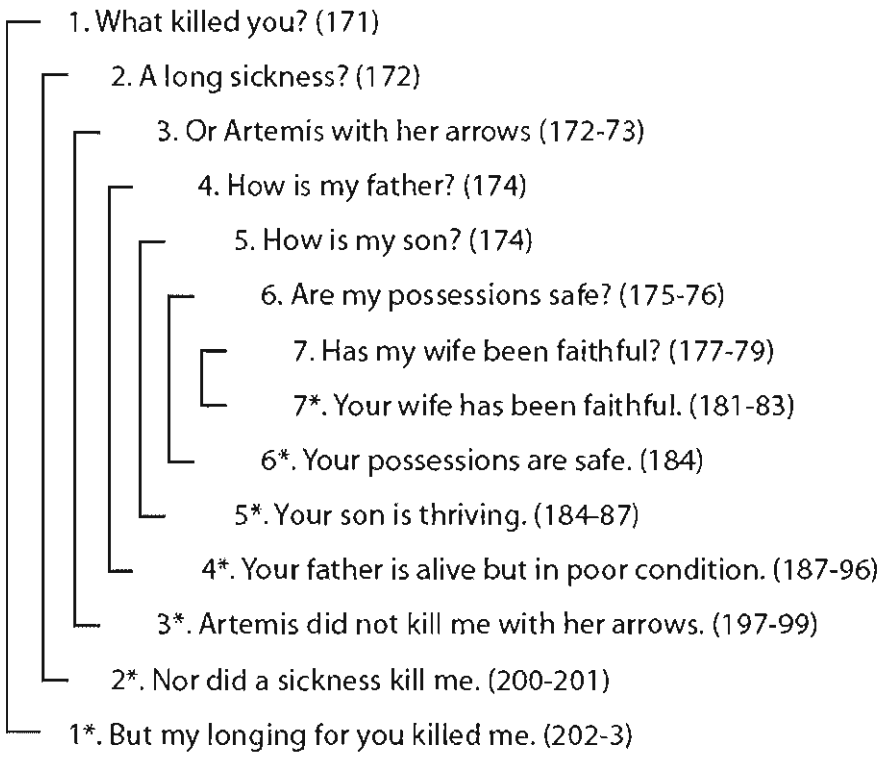

Figure 10.12. A long X-structure from the *Odyssey* (11.171–203), summarized in Reece (1995, 213).

The underlying spatial form would account for, among other things, the series of questions and answers in the epics, where questions are answered in exact reverse order, as in the long X consisting of Odysseus's questions to his mother, Anticleia, in Hades (figure 10.12; the numbers refer to verses; formatting follows the standard notation for denoting X).). However, X/RC is not just mnemotechnics. Referring to its use in epic, Steve Reece calls it "perhaps the most important structuring device of oral narrative, building bridges between the many components of the larger poem . . . weaving the digressionary material into the larger fabric of the narrative." Though X/RC may have begun as an unconscious mechanism and "survived for its mnemonic and tectonic value," it soon turned into "an aesthetic principle as well, becoming a desirable and expected pattern of oral narrative" (Reece 1995, 220). Few scholars would disagree with the statement that X/RC is one of the frequent stylistic features of epic form. However, there is scholarly dispute as to both the extent and the range of the employment of X/RC, as well as whether it is always a device that is consciously applied by the poet or a mere consequence of laws also applying to quotidian orality.[29]

Figure 10.13. RC-type composition, with central figure surrounded by a symmetric arrangement. Attic vase, dated between 600 and 530 BCE. (Staatliche Antikensammlung, Munich.)

How X and RC got into Greek poetic storytelling in the first place is not our concern here. Suffice it to say that though they could have come to Greece from earlier literary traditions, such as Sumero-Akkadian and Ugaritic, where they are also prevalent, they could also have been indigenous, as there are strong indications that X/RC are cognitive universals, either because of their underlying spatial logic or possibly through a relation to natural mythic cycles (Miller 1982, 800; Douglas 2007, 41). Spatial analogues, and thus also, possibly, inspirations of X/RC in Greek poetry can be seen in two different art forms: (1) the choral practices of lyric poetry, and more specifically the circular dances, where a group of dancers changes direction, singing along the way; thus, the dancers would sing verses A, B, C, D, E when going, say, clockwise, and then their "inverses" when going counterclockwise, which would go in the order E*, D*, C*, B*, A*;[30] or (2) archaic architecture and vase painting, with vase painting's preference for symmetric structures, often with a unique central element as the focus. In the vases in figures 10.13 and 10.14, we have scenes that are symmetrically set around a central element. Thus, reading the composition in figure 10.14 from left to right, we get helmet–hero–board game–hero–helmet, or an RC structure of the form A-B-C-B*-A*. Compositions such as these, in Geometric vases, appear as early as the eighth century BCE.

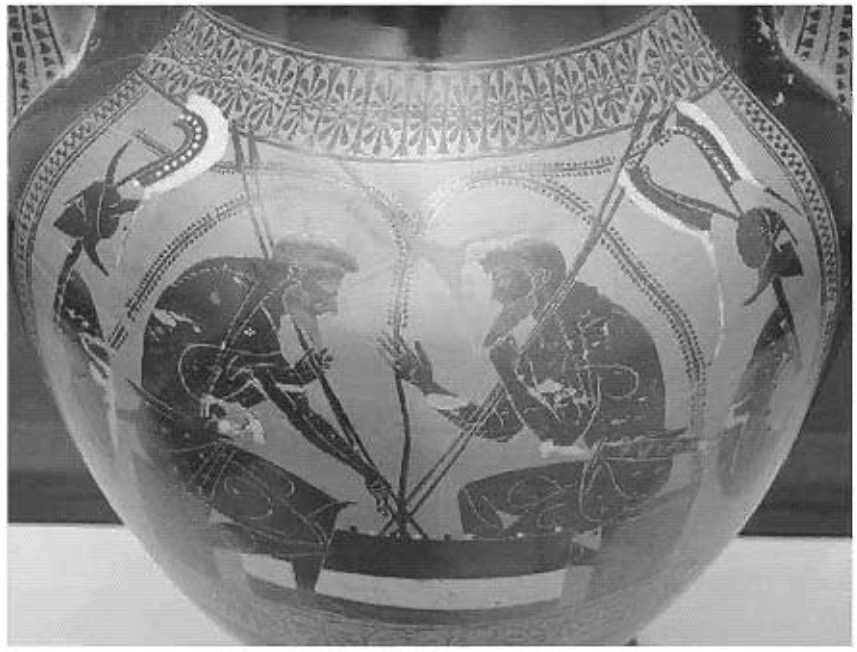

Figure 10.14. Achilles playing dice with Aias. Attic vase of the Leagros group, c. 500 BCE. Staatliche Antikensammlung, Munich.

Bassett was one of the first to suspect that there is more than aesthetic or abstract structural value in X/RC: "We . . . may ask how far the chiastic order was determined by . . . [the] *arrangement of ideas* . . . and possibly the *element of surprise which sharpens the attention of the listener*. . . . [Potentially relevant, too, is] the psychological factor, the advantage of *using one idea to suggest another*, and thus to *make the thought continuous*" (1920, 59; my italics). If he were writing today, Bassett would have probably used the word "cognitive" instead of "psychological." The importance of Bassett's notion of *surprise* cannot be exaggerated, especially when coupled with *continuity*, a quality that seems almost antithetical to it; but real understanding—by human beings, not machines—often depends on sudden leaps of the imagination that unexpectedly bring together what was previously disjoint. This is a concept at least as old as Plato's *exaiphnês*, the "suddenly" in which the forms (*eidoi* or "Platonic Ideas") are revealed to the soul. Its involvement with X/RC shows us that its use is much older.

John Welch, a scholar who has extensively studied the use of chiasmus, mostly in biblical literature, believes that one of the functions of an X is to structure thought patterns and whole literary units (1981). Rodolphe Gasché, after calling X "an originary form of thought, of dianoia," makes one of the very few concrete references I have found in the literature on how chiastic structure actually relates to the higher cognitive process we call thinking: "Originally, as a form, as the form of thought, chiasm is what allows *oppositions to be bound into unity* in the first place" (quoted in Warminski 1987, ix–xxi; italics in original). In a particularly insightful paper, Paul Friedrich proposes X as one of the elements of what he calls *lyric epiphany*. He writes of X as creating "an illusion . . . of a synchronic, monocular vision of an absolute aesthetic truth—usually with a radical closure" (2001, 218), and goes on to speak of X being so pervasive "in archaic Greece to warrant our speaking of a chiasmic mind, or mental set, or even worldview" (240).

It is precisely this side of the function of X/RC, whether expressed as Bassett's "element of surprise" or Friedrich's "illusion of a synchronic, monocular vision," that could be at the root of the exaptation process we are examining. In other words, while X/RC had become prevalent in archaic literature for its combination of mnemotechnic usefulness and a pleasing rhythm of symmetry, it later acquired new uses in rhetoric because of its capacity to give continuity and unity to sequences of utterances that were not connected with either the rationale of action

representation or mere aesthetic delight, in other words the two dominant rationales for putting sentences together in quotidian narrative and poetry until that time. The new need, guiding the exaptation, was persuasion, the aim of rhetoric.

For an example of this seemingly uncanny—because explainable only by recourse to the cognitive capacities of the human mind—power of X, we may turn to Gorgias's *Defense of Palamedes*, another exhibition piece on a fictional case. Palamedes is defending himself against Odysseus' accusation of treason. After charging Odysseus with having said about him opposite things (*enantiotata*), which is to Gorgias a very clear criterion of untruthfulness (Spatharas 2001, 398–99), the rhetor, as Palamedes, sums up his argument with an X that combines his earlier statements to deal Odysseus' arguments a fatal blow: "If I am wise, I did not err. If I erred, I am not wise" (9). Both periods, A-B and B*-A*, are, in themselves, unimpressive. Yet their combination in this A-B-B*-A* form, with "wise" and "err" as pivots, has a power to impress that seems irreducible to a purely formal argument. In this sense, the effect of X/RC, like that of other poetic tricks, such as rhyming, is like that of a joke, whose effect cannot be explained as a mere sum of its parts. It works as a cognitive gestalt: when you try to take it apart, or explain it, it gains in clarity only what it loses in impact.

Rather than attempt a historical description of how a narrative form grew into a persuasive one, through use of X/RC, I will present a brief typology of X/RC in some of the early classical forensic speeches, in the last decades of the fifth century BCE and the early ones of the fourth. These forms are clear evidence that, by this point in time, the practices of poetic storytelling are beginning to be employed not just for telling stories but for persuading an audience of citizens that one narrative is more convincing than another. We cannot know to what extent this transformation was consciously effected. But Socrates' view of the rhetors and the sophists as confidence-tricksters who could fool their audience into believing pretty much anything through verbal prestidigitation could well be motivated by the belief that they were merely, though very artfully, playing with words or, in our own modern parlance, "putting the signifier over the signified"—which, in a very literal sense, is exactly what they were doing. If they were not highly successful in their trickery, one thinks, Socrates would not have made such an issue of it.

4.2. Ring-Composition as Cognitive Tool

The proof part of a forensic speech works with the narrative material in the narration, which it seeks to enhance both positively and negatively: both by making it more plausible in itself and defending it against alternative accounts. The two forms of artistic *pistis* most used are the *paradeigma*, described by Aristotle as "rhetorical induction," and the *enthymeme*, which he calls "rhetorical syllogism." These methods of argumentation are intimately related with X and RC; in fact, they are elementary cases of these forms.

The rhetorical assimilation of the Homeric use of *paradeigma* provides a good example of the refitting of poetic storytelling practices to persuasive purposes. In Homer, the *paradeigma* typically takes the form of an appeal to a mini-story, usually mythological, to justify a certain behavior. What makes this particularly pertinent to our argument is that the standard framing device for *paradeigma* in Homer is RC. In the example in figure 10.15, from the last book of the *Iliad*, Achilles, addressing Priam, proposes that they set aside, for a while, their mourning for Patroclus and Hector, respectively, and have dinner (the numbers refer to verses; formatting follows the standard notation for denoting RC). A simple (i.e., merely containing the necessary information) appeal to an example would exhaust itself in 1-2-3; 2* and 3* do not add anything new. Definitely, one of the main uses of the full RC form in epic is to blend digressions more naturally into the main narrative.[31] But as it happens, this arrangement also reinforces the double injunction (1-1*) through analogy with the old story in a way that is both

continuous and, because of its symmetry, aesthetically pleasing—and thus more convincing.

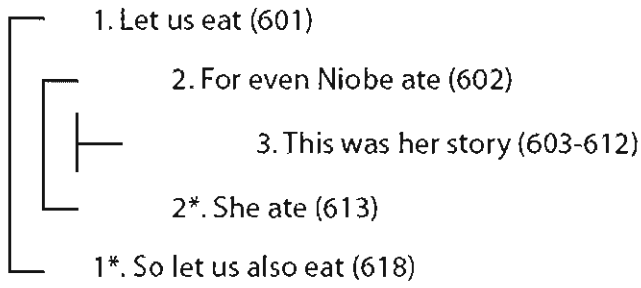

Figure 10.15. *Iliad* 24.601–18; abridged from Willcock (1964, 142).

4.2.1. RC-type Rhetorical Argument

Fifth-century BCE rhetoric borrows this structural model almost exactly, though the aim of the RC-patterning occasionally changes, from injunction to demonstration. This change happens when an action phrase ("x did so") becomes a state-of-affairs phrase ("x is so"), the action verb morphing into the copula "is." The central element of the RC can still be a mini-story, in Homeric style, but it is more often a *gnômê*, as in the speech "On the Murder of Eratosthenes" by Lysias (ca. 400 BCE). A husband is defending himself for having murdered a man, surprised in flagrante delicto with his adulterous wife. In our example, he is appealing to the fact that the punishment for adultery by Athenian law is death and thus, by killing Eratosthenes, he has done nothing more than apply it. (Bracketed phrases in the speech in figure 10.16 are added for clarity, while pivots are marked by bold italics in 1-1* and underlined in 2-2*.) In a strictly logical sense, the sequence 1-2-3 contains all the necessary information: purely in terms of information given, 2* is a mere repetition, while 1* is a transformation of the statement of fact in 1, to an appeal. But it is 2*, which completes the RC, that creates a continuous link from the defendant's action to the court's decision. Thus, it is not just the undeniable truth of 3 that makes the appeal much stronger than a mere "please acquit me." It is also the connectivity of the pattern, which leads from a state-of-affairs sentence (1) to appeal (1*) via general rule (3) in a way which appears seemless and continuous—and thus more convincing.

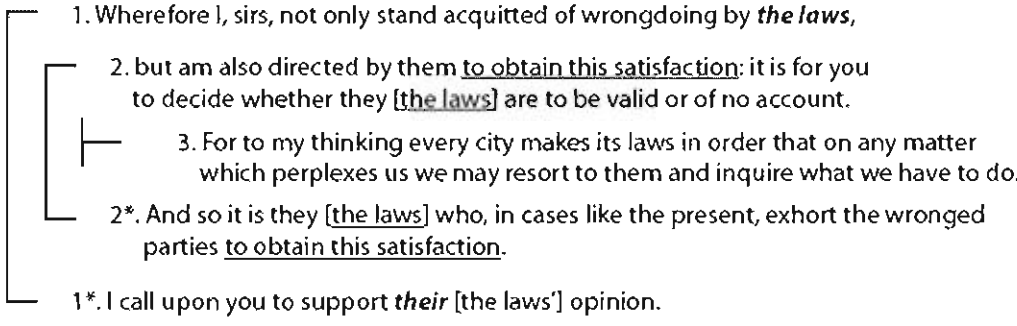

Figure 10.16. From *On the Murder of Eratosthenes* (Lamb 2006, 21).

One might argue that in this case, the RC argument is more a rhetorical formal trick than application of a logical rule. But that is exactly the point: at this historical stage, *there is no clear demarcation of the two*. And it is only through an examination of this intermediate stage that the roots in poetic story telling of (what is later seen as) logic become apparent.

4.3. Chiasmus as Cognitive Tool

G. E. R. Lloyd (1966) has identified *analogy* and *polarity* as two major conceptual categories of archaic thought, both having roots in mythology. The RC pattern we just described is based on highlighting analogy (either with a paradigmatic story or general rule), whereas the two basic X-type arguments I have found in extant rhetorical texts of the fifth and fourth centuries BCE—and of course there may be others—depend on polarities.

4.3.1. X-type 1 Rhetorical Argument

For an example of the first X-type argument, let us look again at the *Encomium of Helen*, to see how Gorgias deals with the first item in his division, described earlier, i.e., the possibility that Helen's abduction was decided by the gods (figure 10.17). A passage that seems ponderous and repetitious to the modern mind—these are standard accusations against Gorgianic style—reveals a perfect structure if we see it as the extended X it is: the contrasting elements in 1 and 1* (underlined) and the identical elements in 2 and 2* (bold italics) are the pivots for the outer parallelism, while the central pair 3-3* (bold) forms a chiastic *gnômê*, in pure A-B-B*-A* atomic X form (its A-pivot in bold, its B-pivot in italics), that the audience can accept as true without further argument.

1. Now if through the first [will of the gods or decree of fate], it is right for the <u>responsible one</u> to be held responsible

2. for the ***gods'*** predetermination cannot be hindered by human premeditation.

3. For it is the nature of things, not for the **strong** to be hindered by the *weak,*

3*. but for the *weaker* to be ruled and drawn by the **stronger**.

2*. ***God*** is a stronger force than man in might and in wit and in other ways.

1*. If then one must place blame on fate and on a god, one must free <u>Helen [who is not responsible]</u> from disgrace.

Figure 10.17. *Encomium of Helen* (my translation; Greek text in Diels-Kranz 1966, 289–90).

In a sense, this is a case of pure argument, without narrative or poetic elements. However, it is more accurate to see it as a mixture of both argumentative and narrative elements, joined through a poetic technique. There are three reasons for this: (a) The outer elements 1 and 1* contain injunctions, connecting what are basically action sentences, the very stuff of narrative: from "hold the right person responsible" to "acquit Helen." (b) Elements 2 and 3 begin with "for," thus indicating a logical type of connection from the previous phrases, 1 and 2, respectively. Yet 2* and 1* are without such direct links from 3 and 2*: their existence and position are pattern-dictated; they are there in accordance with the overarching X form, a poetic form, which is especially apparent in the presence of 2*, a statement otherwise logically redundant. Finally, (c) it is too early in intellectual history to speak of "pure argument." It is a half century before the theory of syllogisms is articulated, by Aristotle. In fact, at the time the *Encomium* is written, the supreme instances of argumentation are in the Attic tragedies and the speeches of the orators. In these, argumentation mainly consists of weaving verbal patterns, using traditional poetic tools to serve a new purpose: not to spin tales but to compare them, to contest others, with the aim of persuasion. Clearly, we have here a case of the form being stronger than the content: the poetic effect of X adds persuasive power to the construction, a cognitive reinforcement of meaning. To call it "logical" is to give it the name of a way of thinking that is still, in the last decades of the fifth century BCE, in the process of being developed.

4.3.2. X-type 2 Rhetorical Argument

The second X-type argument prevalent in rhetoric is also guided by a polarity, in this case the specific form of *contradiction*. The central chiastic quadruple—which can be lodged in at the centre of a much larger X (figure 10.21)—is here formed by a pair of action phrases, the first two of which form a hypothesis: "If *a* happens (A)," "then *b* happens (B)." The third (B*) is a denial of B, either from the narration ("*b* did not happen") or based on narrative probability ("*b* could not have happened"). The fourth, A*, completes the X pattern symmetrically, performing the work of what is later formalized as logical conclusion or *sumperasma* ("*a* did not happen" or "could not happen").

1. Nobody who was exposing his life to a grave risk [the **malefactors**]
    2. Would forgo the prize (i.e. the cloak) when it was within his grasp.
    2*. And the victims were found wearing their cloaks.
[1*. **Malefactors** are not likely to have murdered him.]

Figure 10.18. Antiphon's *First Tetralogy* argument, in full logical form.

The following is a simple example of this type from Antiphon's *First Tetralogy*, guided by the division which we examined in 3.2.2.d. In this passage, the orator is trying to rule out one of the counternarratives to his narration (115.4; transl. K. J. Maidment):

> Malefactors are not likely to have murdered him, as nobody who was exposing his life to a very grave risk would forgo the prize when it was securely within his grasp; and the victims were found still wearing their cloaks.

There is something strange about this simple X-like pattern: 1* has been moved to the top. In proper logical form, the X would have read as shown in figure 10.18. In this arrangement it conforms to Aristotle's conception of a syllogism as *ex anangês*, "of necessity" (*Pr. Anal.* 86.1 and elsewhere), in the sense that if we accept 1, 2, and 2* as true, then 1* is a necessary conclusion. But it also exhibits the characteristic effect of the X, namely, it "tightens and closes; there is an element of inevitability as exit replays introitus" (Friedrich 2001, 241).

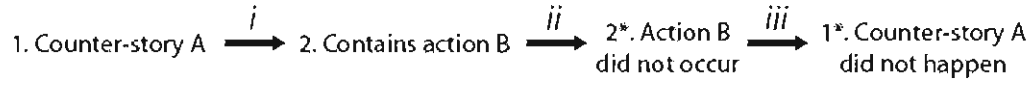

Figure 10.19. X-type 2 argument in generic form.

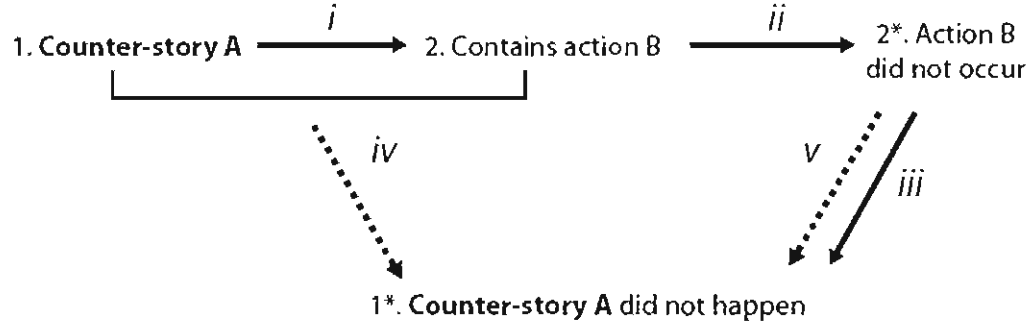

Figure 10.20. The nonlinear expansion of the X-type 2 argument.

Let us write this type of argument in its generic form, the arrows in figure 10.19 marking the

links from one phrase to the next. Notice that links *i* and *ii* make full sense as linear arrangements: *i* joins the two parts of an action sentence, the second completing the first, and *ii* is an instance of the common figure of *anadiplosis*, in which the last word of a sentence is repeated at the start of the next, a figure, incidentally, which appears at the center of every X. But link *iii* is not justified from the content of the previous phrase: there is no linear rule taking us from 2* to 1*. This transition can be explained only if we map the implicational connections (in dotted arrows in the diagram) in two dimensions, as shown in figure 10.20. In other words, for the implication to hold, we need links *iv* and *v* operating *together*. From this we can see the full power of the X structure: though, read simply as text, it merely concatenates the four phrases in linear order, the underlying X form encodes their relationships in a way that implies their nonlinear connections. In other words, X is an economical way to *linearly encode a nonlinear pattern*. And it is only knowledge of the underlying X structure that allows the actual reordering in the text, that is, 1*-1-2, to be read correctly.

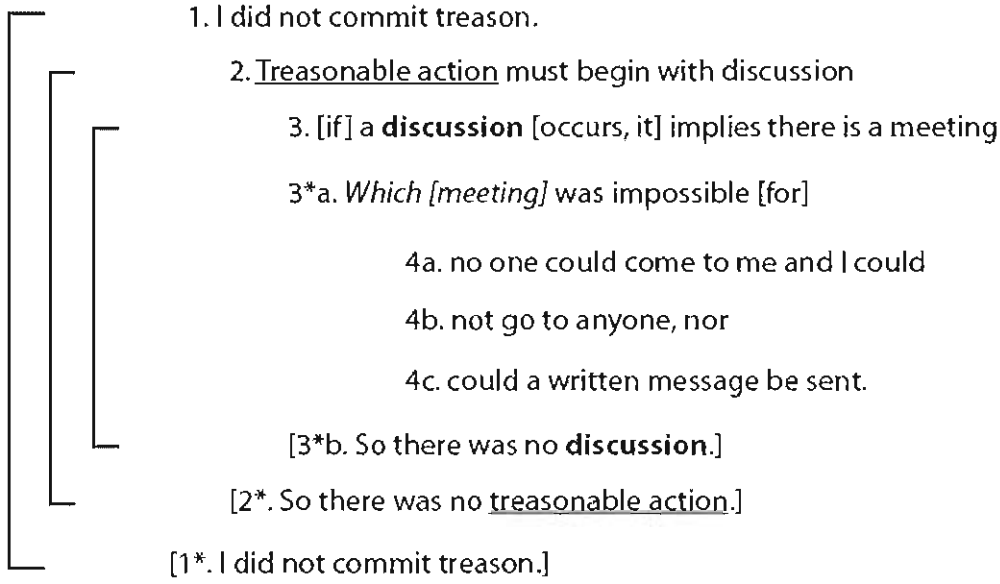

Figure 10.21. Gorgias, *Defense of Palamedes* (my translation; Greek text Diels-Kranz 1966, 295–96).

Now let us look at a longer case of the X-type 2 structure. Figure 10.21 shows the full X form from an example from the *Defense of Palamedes*, another mythological case. In this, additional X-coupling is applied to a basic A*-A-B structure (derived from the full form A-B-B*-A*).[32] Gorgias builds a subnarrative enriching the mythological narration with details provided by knowledge of the real world, via narrative intelligence. Similar to the previous example, the X in the actual speech starts with 1* (the negation of 1: "If I committed treason"). The form of the full X in the speech is 1*-2-3-3*a. The second part of the central chiasmus 3* is here broken down into two parts, 3*a and 3*b, with the three sentences 4a–4c interjected, as additional evidence for 3*a, while 1 as well as 3*b-2*-1* are omitted, as in the smaller form A*-A-B, in Antiphon's speech examined above (as noted, 4a–4c are parenthetical to the main X structure, an extra corroboration of a particular point).

Minds trained in X/RC—as classical minds undoubtedly were—can naturally supply 3*b-1* and effect closure, especially as 1–4 is a cascading question-and-answer pattern, the one leading to the other. In effect, we have in 1-2-3 a series of substitutions of the *demonstrandum*, through a sequence of implied questions: (1) How would I have committed the crime of *treason*? (2) To commit *treason*, shouldn't I have had a *discussion* with someone? (3) Wouldn't this discussion imply a *meeting*? (I have italicized the new *demonstrandum* in each sentence.) Thus the initial hypothesis (1) is transformed, through (2), to one that can more easily be refuted (3) by the

three parallel elements 4a–c.

The long X-type 2 argument has the generic form 1*-2-3- . . . -N-N*, where 1, as well as the sequence (N-1)- . . . -3*-2*-1*, are omitted. The reason why this is actually a chiasmus, despite the lacunae in its form and the transposition of end to beginning, has to do with the particular nature of rhetorical proofs, which are almost always gappy in this exact way: many obvious things are left unsaid. To be rhetorically persuasive, a concatenation of phrases need only be complete enough to lead a listener over its many gaps, to the underlying complete form. Aristotle, stresses that in rhetoric, "the conclusion must neither be drawn from too far back nor should it include all the steps of the argument," as the first situation "causes obscurity," while the second "is simply a waste of words because it states much that is obvious" (*Rhetoric* 1395b; my translation).

5. Proof in Mathematics

My partly externalist account of the origins of deductive mathematical proof in classical Greece is based on the view that culture-specific factors played a crucial part in its development. As G. E. R. Lloyd's comparative studies of the development of Greek and Chinese mathematics show, "there is nothing inevitable about the way in which [mathematics] developed, and [its] international modern character should not masks [its] very divergent early manifestations" in different cultures (1996, 225). Greek mathematics cannot be an exception to the ubiquitous process in cultural development that new practices are partly developed by applying some of the tools and techniques of older, preexisting ones.

In section 5, I extend my previous arguments about rhetoric to show that both the macrostructure and the microstructure of the first proofs in Greek deductive mathematics were crucially affected by the dominant—within the adversarial cultural context of the Greek polis—practice of forensic rhetoric, as this was shaped, as we saw, under the influence of the cognitive mechanisms of narrativity and the forms of poetic storytelling. At the level of macrostructure, I propose that the form of the Attic forensic speech provided to a large extent the template for the shape of mathematical proof. At the level of microstructure, I show that X/RC constructions, similar to those that played such a central role in the development of demonstration in Greek rhetoric, are among the main tools of the new practice of mathematical deduction. As Euclid's deductive proofs are the first eminently worthy of the name in recorded intellectual history, describing the presence in them of forms originally appearing in poetic storytelling, transmitted to mathematics at least partly via rhetoric, is essentially tantamount to finding the missing link in the cognitive history of mathematics, from no proof to proof: our streetcar will have reached its final stop.

5.1. Domain-Related Influences on the Development of Greek Geometry

Even schoolchildren know—at least they did a couple of decades ago—that the basic tools of Greek mathematics were the ruler and the compass. In vases of the early geometric age (ca. 900 BCE) we see for the first time the tiny characteristic pin mark in the center of perfectly drawn circles, evidence of the use of some kind of circle-drawing tool. Yet these circles were not drawn with a compass, but with what archaeologists call a "pivoted brush," consisting of little brushes attached by a vertical piece of wood and moved around a nail at one end of the wood (Papadopoulos, Vedder, and Schreiber 1998). Yet the discovery and general availability of the kind of compass that could actually affect mathematical thinking, i.e., the adjustable double-edged compass, came much later in the archaic period, probably no earlier than the last decades of the sixth century BCE.

The role of this invention has not received the attention it deserves. Use of the adjustable straight-edged compass followed the fate of many tools: a tool is invented for a certain use, but as its users become more sophisticated in this, other uses are invented for it. Thus, though it is quite likely that the adjustable compass was invented for marking circles of various sizes on surfaces, mostly in pottery, it soon became clear that it was an ideal tool for reproducing a certain specific length. The observation that all points on the circumference of a circle are equidistant from the center, as well as Euclid's second postulate, namely, that given a point and a straight line, one can draw a circle with the point as center and the line as radius, seem to be the kind of practical knowledge an experienced compass user would naturally possess, long before there was any concept of theorems and axioms. And there were many such practiced compass users in the Classical Age, people possessing what we can call *compass literacy*.

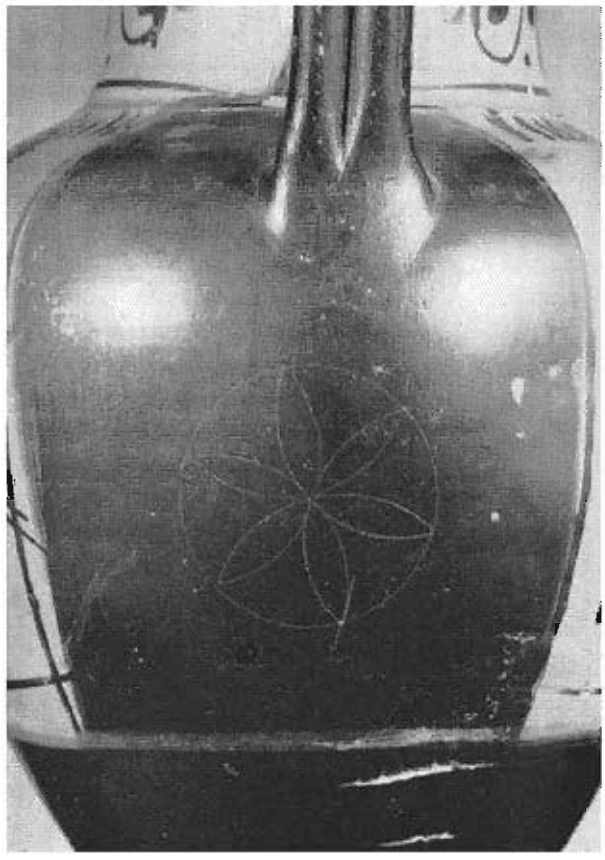

Figure 10.22. Glazed-over drawing on a doubleen amphora, attributed to the school of the Athenian painter now referred-to as the "Edinburgh Painter," ca. 500–490 BCE. (From Noble, *The Techniques of Painted Attic Pottery*, 115).

One particular piece of early evidence of compass literacy is worth mentioning, as it shows the kind of geometric achievements that could be achieved by vase painters, without the employment of formalized geometric knowledge. In figure 10.22 we see, hidden under the black glazing on an early fifth-century doubleen amphora, an erased marking of overlapping circles. Noble (1988:54) thinks that its position, under the handle, indicates that this was a sketch, possibly for a shield. This is the common way to create an asterisk-shaped decorative form on shields, on late sixth- and early fifth-century BCE black figure vases; we use this particular one, as the underlying pattern is so perfectly visible. This sketch, made two centuries before Euclid's *Elements*, by a compass-literate vase painter, employs an advanced version of the technique that is used to construct an equilateral triangle on a given line segment with compass and ruler in *Elements* proposition 1.1, as indicated in figure 10.23. (The technique used on this vase is also sufficient to create a regular hexagon on a given side, as can be seen in

figure 10.24). Euclid's proof of proposition 1.1 depends on the fact that all points on the circumference of a circle are equidistant from the center—there seems to be no better way than actually drawing the circle with a compass to physically realize this—as well as the first of what Euclid terms a "common notion" (*koinê ennoia*), or "things that are equal to a third thing are equal to one another."

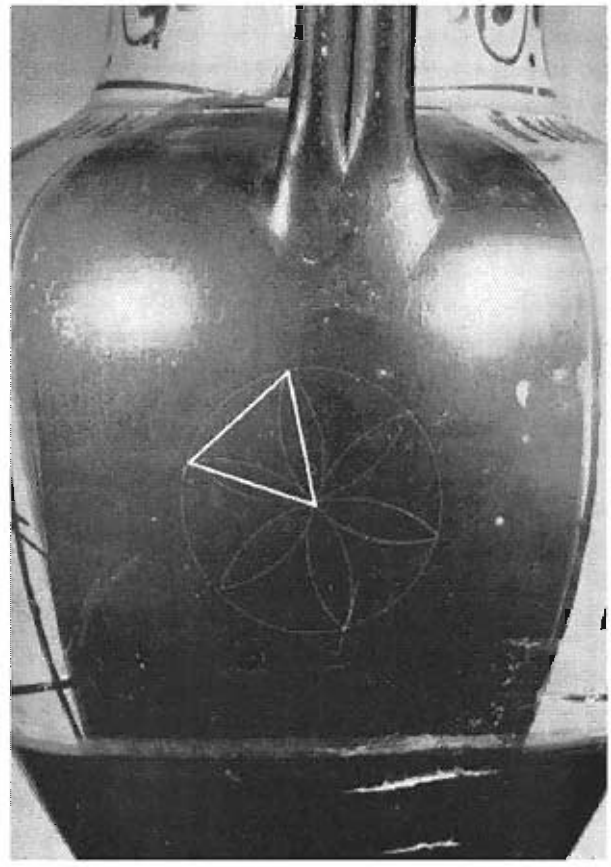

Figure 10.23. The way an equilateral triangle results from the underlying drawing in figure 10.22, also in accordance with the technique used in *Elements* I.1.

It is important to note that the passage from the habits of the Greek craftsman (who drew this particular sketch on the amphora) to the habits of the Greek scholar (who proved, two centuries later, proposition 1.1) need not be as clearly marked as an older, idealized version of the development of higher thought processes might imply.[33] In the fifth and fourth centuries BCE, there could well have been in Greece many such cultural centaurs, in other words people who combined the skills of the craftsman and the thinker. The first Greek mathematicians could well have belonged to this particular species. For though Greek deductive mathematics is famously "abstract," "philosophical," "nonapplied," and "impractical"—all clichés of its older historiography—there is no reason why it couldn't have developed partly from knowledge of a craft or the opportunities given by the new tool, the adjustable double-edged compass used by vase painters in the latter part of the sixth century BCE. From this particular viewpoint, a lot of the abstraction—as it is usually considered—encoded in Euclid's axioms can be seen as no more than a setting down on papyrus of the basic grammatical rules of a craft involving compass literacy, for the purpose of translating its results from its original form of communication, in the live, oral interaction between peers, using a writing surface on which they can draw with ruler and compass, and also refer-to through deictic communication, into the language of written communication. The first three axioms, especially, encode the necessary arrangements for the mathematician to be able to effect this transition, from orally-delivered to written theorems, without losing information or weakening the capacity to convince.

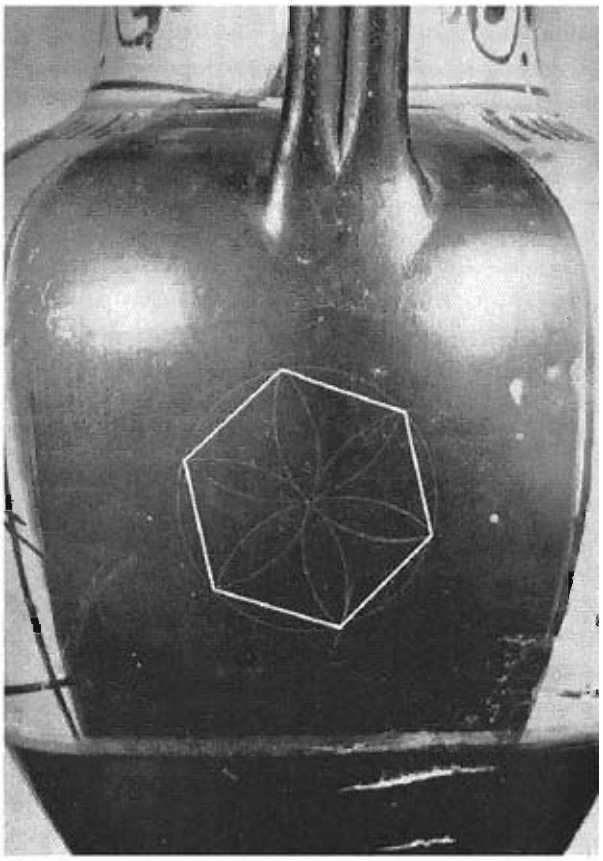

Figure 10.24. The way a regular hexagon results from the underlying drawing in figure 10.22.

Apart from vase painting, early geometric practitioners could have applied their compass literacy to architecture and town planning, already a flourishing practice in the late Archaic Period.[34] The Greek architectural practice of building from small three-dimensional models (Coulton 1977) set great demands for an abstract understanding of relations of proportion. Though we have no written evidence substantiating this provenance, the quantity of theorems that conclude that four line segments, or four areas, are unified in a particular quadruple of proportions of the form "*AB* is to *CD* as *EF* is to *GH*" could well have its roots in the needs and opportunities created by this particular architectural technique: the application of proportional rules in architecture requires absolute certainty, which can only be reached through deductive proof. In the artistic craft of the vase painter, the proof of the pudding is in the eating (i.e., if a geometric construction *looks* right, for all aesthetic purposes it *is* right); but in creating large buildings by applying transformations of proportion, in scale, to much smaller ones, it is necessary to know that one is absolutely right, as minute inexactitudes of measurement in the model can have gross consequences when multiplied by big factors.

5.2. A Bridge over Troubled Water: From the Language of Narrative to the Language of Demonstration

The first deductive mathematical proofs were probably disseminated in the live interaction of small groups of compass-literate people, early "experts" of sorts, chatting around a writing surface (sand, wax tablet, or papyrus) on which diagrams could be drawn. And though the invention of new theorems and proofs is traditionally considered a solitary affair, these interactions of experts could also have functioned, on occasion, for research to generate or perfect new results. Gesturing and deictic language ("this triangle," "if I draw this line from here

to here") played a big part in these expert interactions in which, gradually, a *standard vocabulary* and *formulaic expressions* were developed (Netz 1999a, 127–67). Such linguistic forms are possible because of the restricted universe of reference of geometry—points, lines, shapes, and the actions of drawing lines and circumferences and comparing them[35]—and result from a developing tradition of communication within an elite caste of experts.[36]

Before I proceed, I propose a simple thought experiment as a basic context for the discussion that follows. Let us think of the (written) proof of a Greek theorem as the *record of the goal-oriented activity of a mathematician in the process of trying to show that a certain geometric statement is true.* In other words, let us think of it not as "abstract thinking" but as the *symbolic representation of the action of proving* something. By our earlier definition, this makes it, precisely, a narrative. One might object that this is a trick, that it is in fact the verb "proving" that makes this representation of action a proof, and that to call it a narrative does not in any substantial way change the nature of a proof, from what we have always thought it to be: a proof is a proof is a proof—*not*, in any interesting way, a narrative. As I shall argue, however, this view of proof provides a good starting point for seeing it in a new context. In what follows, I am keeping in mind that a proof is the representation of particular actions of oral-environment "proving," i.e., pointing, drawing lines or circles, and comparing, etc. This process can include verbal utterances.

In looking at proofs as narratives, I apply what we know about narratives. A proof—we are now looking exclusively at early Greek deductive theorems—uses the same three types of atomic sentences as all narratives: action sentences (e.g., "I draw this line"), state-of-affairs sentences ("this triangle is similar to this"), and *gnômai*-like expressions of general rules ("things equal to the same thing are equal to one another"). It should be noted, however, that the relative percentages of the three types of sentences occurring in proofs-viewed-as-narratives are different from those we find in other narratives. But, more important, these percentages vary from theorem to theorem and also from part to part of a theorem. We shall look at two examples to illustrate this variation. In these, I will follow the following conventions: (1) atomic sentences are separated by //, (2) action sentences are <u>underlined</u>, state-of-affairs sentences are in regular type, and general rules in **bold**; and (3) working within our thought experiment, I will sometimes translate the abstract, impersonal diction of Euclid into a speaking mathematician's first-person account.

The first, thus modified sample is from proposition 1.9 of Euclid's *Elements*. In this and subsequent Euclidean excerpts, the bracketed text, containing references to earlier propositions or comments, is the translator's. (The following and all further translations of Euclid's original text are from Fitzpatrick 2005). Yet we retain them here because in an interactive context of presentation and discussion of proofs, we can imagine some of these coming up in answer to questions such as "but how can you do this?," which bring up, as justification, previously proved results, definitions, common notions, or axioms.

> BAC is the given rectilinear angle. // So it is required <u>to cut it in half.</u> // <u>I take a point, D, randomly on AB, // I take an AE, equal to AD, from AC, [using the method given in **Prop. 1.3**]</u>. // <u>Then I join D with E with my ruler.</u> // <u>Then I create an equilateral triangle DEF, with DE as one of its sides, [using the technique given in **Prop. 1.1**]</u>.[37] // <u>Now I join A and F.</u> // <u>Now I say that I can prove to you</u> that the angle BAC has been cut in half by the straight-line AF.

This sample of Euclidean text fits almost perfectly the definition of narrative as the representation of action, with our geometer as main character. The percentages of the various types of sentences are the same as in normal, quotidian narrative: action sentences predominate, sentences describing the various steps in a purposeful human action, performed toward the

fulfillment of a goal. The sample contains just two state-of-affairs sentences, the first of which is deictic ("this is equal to that") and the second stating an atemporal fact about it, namely, that the two parts of angle *BAC* are equal.[38]

The next sample is also from the *Elements* (proposition 3.7) but the percentages of sentences are very different, with only one sentence describing action, two general rules, and the rest state-of-affairs sentences:

> For let BE, CE, and GE have been joined. // And since **for every triangle (any) two sides are greater than the remaining (side) [Prop. 1.20]** // EB and EF is thus greater than BF. // And AE (is) equal to BE [thus, BE and EF is equal to AF]. // Thus, AF (is) greater than BF. // Again, since BE is equal to CE, and FE (is) common, the two (straight-lines) BE, EF are equal to the two (straight-lines) CE, EF (respectively). // But, angle BEF (is) also greater than angle CEF. // Thus, the base BF is greater than the base CF. // Thus, the base BF is greater than the base CF **[Prop. 1.24]**. // So, for the same (reasons), CF is also greater than FG.

These analogies are radically different from those in a normal narrative. Thus, either this part of the *Elements* is not in any meaningful way a narrative or we have chanced on a section of narrative—such as a part of an epic where the poet recites a list or describes an object—that is not at all typical of the form.[39]

Jumping ahead in my argument for a moment, I will say that the reason for the difference in the two samples lies in the fact that they belong to two different parts of the theorem, the former to what ancient authors call the *construction* (*kataskeuê*), the latter to the *proof (apodeixis)*.

5.3. The Form of Theorems: The Forensic Template

Before we describe the generic template of Greek theorems, we must remember that Euclidean propositions are of two kinds: those that demand that a geometric object having certain properties be *constructed*, and those that demand that a certain geometric property, such as equality, inequality, proportion—these three cover the greatest percentage of such theorems—be shown to *be true*, within certain particular constraints. Theorems of the first type end with the statement "which is what was required to construct" (*hoper edei poiêsai*), while those of the second end with "which is what was required to show" (*hoper edei deixai*). The word *deixai* in the latter is the same as appears in the word for proof, *apodeixis*, a composite of the preposition *apo-* (from) and the verb *deiknumi*, which means "to show" or, more specifically, "to point out." Thus, the original meaning of the composite *apodeixis* is something akin to "to show something, from something else." The "something else" could indicate a diagram, a previously established truth, or both.

The origin of the verbal material with which Greek theorems are made, that is, the action sentences describing geometric operations, as well as the state-of-affairs sentences indicating properties (in the specific case) or rules (in the general case), at least partly goes back to practices such as drawing by compass-literate painters, architectural model making, and so on. But when we try to understand the rationale and the methodology of actually proving that a certain geometric statement is true, we have to turn to the cultural influence of rhetoric—and thus also, indirectly, of poetic storytelling. Again, this is effected at two levels, those of macrostructure and microstructure.

To begin with, Greek geometry, like Greek rhetoric, resembles poetic storytelling in constructing its theorems according to solid patterns, or *templates*. By the middle of the fourth century BCE—if not even earlier—a standard template for the geometric theorem has emerged.

With occasional variations, this is it[40]:

1. The *enunciation* (*protasis*), where a statement is given in a general form.
2. The *setting out* (*ekthesis*), where the general statement described in the *protasis* is made symbolically concrete, or instantiated, while maintaining its generality, by reference to a lettered diagram.
3. The *specification* (*diorismos*),[41] where the statement of the enunciation is made in the terms of the setting out, that is, where the actual goal is defined.
4. The *construction* (*kataskeuê*), where additional points, lines, or shapes that will be useful in the proof are constructed.
5. The *proof* (*apodeixis*), proving the statement, as set out in 2 and 3, also using the objects constructed in 4.
6. The *conclusion* (*sumperasma*), which repeats the statement, in the abstract phrasing of 1.

We can place the various parts of the mathematical proof in two classes, with respect to their structural similarity to the basic form of narrative: (1) the *protasis, ekthesis-diorismos, apodeixis*, and *sumperasma*, which can be less or more like a narrative, depending on the high or low (respectively) percentage of state-of-affairs sentences they contain; these are more narrative-like in construction proofs and less so in the others; and (2) the *kataskeuê*, which always has a high degree of similarity to narrative, consisting almost exclusively of sentences representing an action. This difference explains the degree of narrativity of the two samples we examined, the first (from a *kataskeuê*) being more narrative, the second (from an *apodeixis*) less so. If we forget for a moment that the referents of a Greek theorem are in the universe of geometry and look at its contents as formal structure, this template is in direct analogy with that of the forensic speech (table 10-1).[42]

Table 10.1. The standard template of a Greek theorem, according to Proclus (left) and its analogies with the standard form of the forensic speech (middle), as set out in Aristotle's *Rhetoric* (right); the third column indicates the function of each part within the larger form.

Geometric Theorem	Forensic Speech	Function
Protasis	Prologue	Setting out of the claim, in general form.
Ekthesis-diorismos and *kataskeuê*	Narration	Description of the concrete facts.
Apodeixis	Proof	Arguments making the claim stronger, and counter-claims weaker.
Sumperasma	Epilogue	Repetition of claim in general form.

5.3.1. The Plot Thickens: The Form of Proof

A forensic speech aims at showing that a particular set of events happened as narrated by the speaker, and that a certain law applies to judging them. Likewise, the geometric theorem

proceeds to show that the concrete situation set out in the *ekthesis-diorismos* and *kataskeuê* embodies the truth stated in the *protasis*. The analogy becomes stronger if we compare the proof part of the theorem with that of the forensic speech.

In the left column of table 10.2, I use the structurally and functionally similar rhetorical terms for the description of the parts of a geometric proof[43]—in quotes, when used this way—to emphasize their similarity. Thus, apart from the main distinguishing characteristic—that a geometric theorem employs no *êthos* or *pathos* arguments—the macrostructure of proof is the same in both cases: first the problem is broken down to constituent parts that are more amenable to proof, then the prover proceeds by using both nonartistic (*atechnoi* or imported) and artistic (*entechnoi* or invented) proofs to attack the constituent parts one by one. The use of both "nonartistic" (external to the theorem) and "artistic" (internal) proofs, and their distinction, is one of the strongest imports into mathematics from forensic practice. In fact, it is precisely this import that accounts for the strong recursiveness of mathematics and the interrelatedness of many of its propositions (Netz 1999a; Barany 2009; Sialaros 2011). As a rhetor imports material into the speech, often in an adapted form that better suits his purposes, so the mathematician relies on preexisting material, either in the form of first principles (definitions, common notions, axioms) or of previously proved theorems. In proving a mathematical theorem, a Greek mathematician does not reinvent the wheel every time he needs a result but uses preexisting ones at will. To the possible objection that these preexisting results are not always explicitly mentioned in the *Elements* but often implicitly assumed, there are three answers:

Table 10.2. The proof or *apodeixis* part of geometric proof (left), and its analogies with the proof or *pistis* part of a forensic speech; the third column indicates the function of each part within the larger form.

Geometric Theorem	Forensic Speech	Function
"Division"	Division	Breakdown of problem in possible alternatives and (often) counter-alternatives.
"Proof proper"	Proof proper	Strengthening of each one of the possible alternatives set out in the division.
"Inartistic proofs": *Axioms, definitions, common notions, previously proven theorems*	Inartistic proofs: *Testimonies contracts laws*	Proofs given to the speaker from outside the case/theorem.
"Artistic proofs":		
—	Ethos	Proofs created by the speaker using arguments of character.
—	Pathos	Proofs created by the speaker using arguments of emotion.
"Logos"	Logos	Proofs created by the speaker using only the materials given in the *ekthesis-diorismos-kataskeuê*/narration and general principles, existing outside it.

1. In many propositions in Euclid, the nonartistic proofs *are* mentioned.[44] For example, in proposition 1.1 we read: "But things equal to the same thing are also equal to one another," that is, a statement of common notion 1. Common notion 1 is again referred-to in 1.2, and used without being explicitly stated in 1.3. Then, after a ten-proposition hiatus, it is used and referred-to again in 1.13—and then again used without being referred-to in 1.14 and 1.15. This strongly indicates that the stating of a previously existing result, when it is not random or haphazard, depends on the facilitation of the readers: by showing them the trick of using a previous result, in its first uses, and by not burdening them with the obvious, in its subsequent omissions. This is exactly similar to the pattern with other nonartistic proofs, such as definition 3.11 (the eleventh new definition introduced at the heading of book 3): when it is first used, in proposition 3.24, it is repeated explicitly: "Therefore, since segment *ACB* is similar to segment, *ADB*, and *similar segments of circles are those accepting equal angles*, etc." (my italics). But when it is reused, in propositions 3.26, 3.28, and 3.29, it is not repeated. Again, the reason must be its obviousness, as it was so recently mentioned, in 3.24.

2. The reason that determines explicit repetition or its absence in the *Elements* may be related to the exact function of the whole work, which is not known. Was it a textbook, a compendium, or something else? We are accustomed to thinking of the *Elements* as the first attempt in human history at mathematical formalism and rigor.[45] But beyond its striving for precision and clarity, which we find so distinctive, Euclid's style is at least partly determined by the style in which which Greek mathematics was written before him, with proofs "in the ancient style" (*arhaikon êthos*).[46] Like the proofs in that style, Euclid's text partly functions as an *aide-mémoire* (*hupomnêma*), a series of reminders to an expert practitioner of the essential points of a proof.

3. Perhaps most basic of all: nonartistic proofs are not often repeated, after their initial statement, in the forensic speech either. In other words, a witness's statement may arise in the course of the speech, or a law may be mentioned as applying to a particular case. But though subsequent, implicit use is made of these inartistic *pisteis*, the witness's statement or the law is not repeated. So if mathematical proof is indeed following rhetorical practice, as I suggest, the omission of previously-stated, necessary premises, is true to the original model in this too.

This last point is particularly important as it leads us straight to a prevalent misconception about Euclid's style. The source of this misconception is our reading—or, rather, misreading, in this case—of the prime ancient authority on logic, Aristotle. Guided by his writings we often tend to assume that mathematical style, being "purely logical," is built up of syllogisms. By this view, it is very unlike a rhetorical argument, which is mostly constructed through the use of enthymemes, that is, "rhetorical syllogisms" (*Rhetoric* 1.1.1355a6). Of course, Aristotle seems to have signified by *enthumêma* two different though sometimes related things (Madden 1952): either a syllogism the premises of which are based on likelihood and not total certainty (*Prior Analytics* 2.27.70a10; *Rhetoric* 1.1.1355a14–18), or a *syllogismos ellipês*, the "truncated syllogism," (Rabe 1896) in which one of the premises seem to be missing, usually left unstated by the speaker (*Rhetoric* 1.2.1357a16–17).[47] Now, though mathematical argument is definitely not enthymematic in the first sense of the term, i.e., it does *not* use probabilistic premises, it most definitely is in the second sense. The examples we have been discussing are a good case in point: often in Euclid, an obvious premise is not stated—in fact, when it is obvious, it is more often than not absent from the text of the proof. In this sense, then, Greek mathematical style is highly enthymematic and in this sense, too, highly rhetorical.

To recap: Classical Greek mathematicians adopted the template of the forensic speech for writing down their theorems, a template that not only created a context for setting out and proving a certain factual event (the truth of a narrative and the application of a law in the first case; that of a geometric construction or relationship in the second) but also provided the model for the proof part of the theorem: beginning from a concrete construction that delimited the universe of discourse—this *kataskeuê* that we saw is practically indistinguishable from a narrative in all formal aspects of its microstructure—the mathematician proved the theorem through a combination of "nonartistic" (first principles or previously proved) and "artistic" (new, in the theorem) proofs.

As I will show, the similarities become even more marked at the level of microstructure.

5.4. The Structure of Proof: Poetic Mathematics

5.4.1. Easier Than It Looks: Making the Switch from Uncertainty to Certainty

The most important way in which rhetorical proof differs from mathematical is in its reliance on probabilistic rather than absolute premises. This is, of course, understandable, as more often than not the aim of the rhetorical proof is to *increase likelihood* rather than to *achieve certainty*. However, it is crucial to remember that its probabilistic nature has to do with the nature of its premises, both the statements of the witnesses, which are not to be totally trusted, and the statistical-inductive nature of the *gnômai* involved. Yet there is nothing in either the template of the rhetorical speech or the structure of the arguments in the logos-type artistic proofs that is inherently probabilistic: the same tools can produce certainty rather than likelihood when starting from absolute, rather than relative, premises. Thus, for example, while the premise "Socrates is a man" when combined with "all men are mortal" leads to the certain conclusion that "Socrates is mortal," if we combine it with the probabilistic premise "most men fear death," it leads not to "Socrates fears death" but to "Socrates probably fears death"—a probabilistic conclusion actually contradicted by facts, if we believe the *Phaedo*. Yet the underlying formal

structure is the same. In other words, the methodology of logos-type rhetorical proof provides a tool that can be employed to arrive both at the flimsier truths of the courtroom or assembly and —given the right premises—the rock-solid ones of mathematics.

At this point, it must be noted that though the proven truths of mathematics are, in the sense that we use the word, absolute, their linking in chains of logical deduction is not always so. We might rephrase this as: "Not all necessary conditions are also sufficient." In this, deductive linking is very similar to causal linking, its equivalent in the world of narrative, in form and structure. We mentioned earlier (see section 2.2) that the causal influence of an action or state-of-affairs A on an action or state-of-affairs B can range from zero (none) to one (total). Furthermore, if the effect of A on a C is indirect, that is, going through B, which influences C, the total effect is the *product* of the two, A to B times B to C—in this, the calculus of causality uses the mathematics of probability (Pearl 2000). A similar thing happens with implication. If we look at the conceptual universe of geometry as consisting not only of its final products, the well-written theorems, but of the full range of geometric statements, from atomic upward, as well as their implicational connections, then we can see that there are two kinds of implication, an absolute one and a partial one. The absolute ones we can also call *rewrite rules*. Thus, the statement "*ABC* is a right triangle" can be substituted by "the square of the longest side of *ABC* is equal to the sum of the square of the two other sides," and vice versa, without any effect on its truth-value. But the statement "*ABC* is a triangle" does *not* lead directly to "*ABC* is a right triangle" (it is a necessary but not a sufficient condition) unless we also know something else, such as that "BAC is a right angle." In a network of implicational connections between statements, neither "*ABC* is a triangle" nor "*BAC* is a right triangle" leads directly to "*ABC* is a right triangle" (for *BAC* may be an angle in a rectangle) though *both*, taken together, do. A legitimate mathematical proof can consist of both absolute and partial implicational connections. In this sense, a proof can be written in the form of what mathematicians call a *directed graph*. Directon of arrows is important in such an object, and cannot always be reversed. In the series of implications forming a Greek geometric proof, we begin from certain premises and move in ways in which their certainty is preserved, all the way to the final statement.[48] Though many different ways can be found to prove a theorem, the certainty of the final statement is guaranteed by the fact that it is reached through a certainty-preserving tree of statements and links (as in the diagram in figure 10.25), leading—if need be—all the way back to the initial certain premises, the definitions (*horoi*), axioms (*aitêmata*), common notions (*koinai ennoiai*), or previously proved theorems (*protaseis*). Of course, before the clarity and the certainty are constructed, through clear definitions, "obvious" axioms, etc., they are there in *the practice itself*, in the world of the compass-literate geometric practitioner of the late sixth and fifth centuries BCE: it is the well-defined nature and the limited range of the basic concepts and actions, as well as the "obviousness" of their starting points, that give the results of Greek geometry as we see them in the *Elements*, their aura of certainty. In other words, it is precisely the qualities inherent in the nature of the practice that allow the methodology of rhetorical proof—which, dealing with the uncertain material of everyday life, produces probabilistic results—to create, when applied to geometrical objects and relations, certainties.

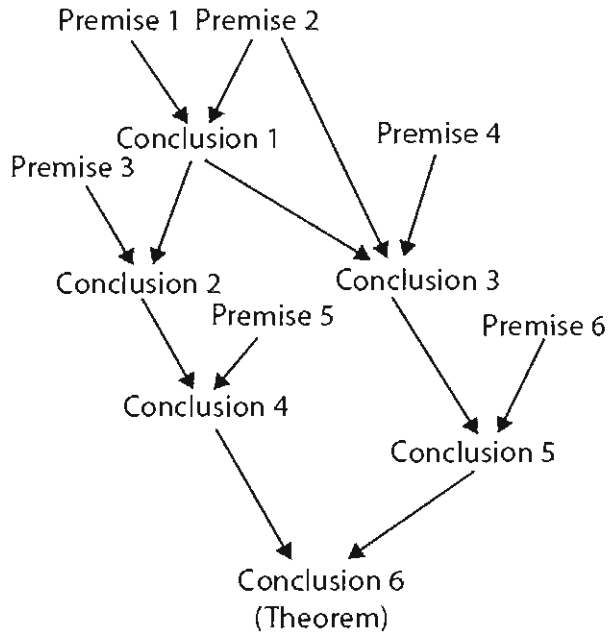

Figure 10.25. The diagrammatic structure of the logical connections between premises and conclusions, intermediate and final, as it appears to a modern mind.

5.4.2. Hints for the Existence of X/RC in Early Greek Mathematics

Partly repeating the arguments made for rhetoric, let me say that the main demand that the new civic needs of the fifth century BCE posed to the microstructure of the emerging methodology of demonstration was how to put together the atomic sentences of everyday speech in a new way, to serve the needs of persuasion. Until then, two basic methods existed for putting sentences together in a stretch of text, oral or written: where the need was representation of action, the method was *narrative*, i.e., temporal and/or causal ordering, structured by the underlying patterns of goal-oriented action; where the needs were those of poetry, either epic, lyric, or dramatic, new *pattern-based* criteria of composition were also combined with the previous ones. Beyond those two methods there existed only the list, the aggregation of elements, usually according to some external order, such as temporal (as in a genealogy) or spatial (as in an inventory).

We saw that the formal analysis of a Greek mathematical proof gives the same kind of atomic sentences that we get in narrative: action ("connect the points," "extends line *AB*," etc.), state-of-affairs ("*ABC* is an isosceles triangle," "as *AB* is to *CD*, so *AB* is to *EF*"), or *gnômê*/rule ("things equal to a third thing are equal to one another," "proposition 2.6," etc.). Each one of these is an element of the conceptual language of a compass-literate practice, refined and augmented through live peer interaction and, later, the demands of writing. The earliest geometers certainly could mimic, in oral speech or writing, a serial, narrative-type ordering of some such sentences to describe a construction ("do this, then do that"); and it is also easy to see how a text might arise describing a spatially ordered structure, following its underlying topography.[49] But mere description was not enough when the aim was to create conviction that a certain relationship holds. To say that the sentences were put together "deductively," that is, in accordance with some inner plan for a proof that would look somewhat like figure 10.25, is merely to bypass the problem, by applying to it a later concept. Logic, as a structured discipline, a self-conscious practice, was still in the process of being discovered. The diagram in figure 10.25 seems to us, today, a reasonable way to map out the methodology of a Euclidean proof, in which the following obtain:

- Information (premises 1–6) is fed into it at various stages, this being either in: (a) items

defined by the *ekthesis* and the *kataskeuê*, or (b) definitions, axioms, or common notions, or (c) previously proved propositions;

- Intermediate conclusions (conclusions 1–5) are reached by the convergence of more than one of these—though simple rewrite rules could also play a part—in which process the conclusions already reached inside the proof can function henceforward as premises; and
- The whole structure converges on the conclusion (6).

However, though figure 10.25 is adequate as a picture of the logical flow of arguments by modern standards, it is less useful as historical representation. The reason is that there is no way to know whether the creators of early proofs could conceive of this abstract flow of logical relationships—in fact, to say that they did is to bypass our central problem of understanding the method through which they learned to put atomic sentences together, to achieve deduction. Our task, thus, is to try to understand what fifth- and fourth-century BCE theorem-provers were doing in a conceptual language that is not too far removed from what they actually knew or could know. For this, we must not employ our own modern models of logical connectivity but look at the proofs themselves: rather than impose a latter-day "logical" structure on them, we must examine their form and extricate from it whatever structures, available to them, their creators seem to have put into them. Examination gives us ample evidence that the mathematical proofs were—as were their equivalents in rhetorical argumentation—infested with chiasmus and ring-composition. I believe that this fact is highly significant for understanding their cognitive history, and thus the appearance of organized, deductive thinking.

In section 4 of this essay, we saw how X/RC was employed in rhetoric to structure sets of narrative-type sentences so as to increase their persuasive power. It is my contention that Greek mathematicians, working with the three types of sentences particular to geometric argument, employed X/RC constructions on principles similar to those used in rhetoric. Before we examine particular cases of proofs for evidence of X/RC, let us look at some more general indications that these forms are central to the emergence of proof, i.e. that proof has an unmistakable X/RC flavor.

1. *At the macrostructure, a proof is enveloped in a three-part X/RC structure.* We saw in section 5.3 that the template of the Greek theorem has strong analogies with that of the forensic speech, with its structural characteristics being clearer since its domain, geometric knowledge, is much better defined. Thus, the function of the outermost pair of parts of a forensic speech, the *prologue-epilogue*, becomes much more focused in geometric proof, framing a hierarchy of gradual entry into the heart of the proof and then a backing-out from it. The proof moves from a *protasis* whose abstract generality makes it all but intractable, to greater specificity, which allows manipulation, in the *ekthesis* and *diorismos*, into the proof itself—and back again, after it, retracing steps: firstly, in a re-statement of the proposition in the specific form, at the end of the proof part, then in the general, in the *sumperasma*. This is the outer structure 1-2-3-3*-2*-1* in figure 10.26, enveloping, in RC-fashion, 4 and 5, the construction and the proof. This outer framing structure, which is particularly clearly demarcated in Euclid's *Elements*,[50] is a strong indication that X/RC is very much on the mind of the mathematical writer.

1. *Protasis*: General statement of the theorem, in terms of situation ("in such and such a case") and result, ("this and this holds").

2. *Ekthesis*: Precise statement of the situation, on a lettered diagram, actual or conceptual.

3. *Diorismos*: Precise statement of the result, in the context of the lettered diagram.

4. Construction

5. *Apodeixis*, which ends in: 3* and 2*

3*. End of *apodeixis*: Precise statement of the result, as was to be proven or constructed.

2*. End of *apodeixis*: restatement of the situation, according to lettered diagram, set out in the ekthesis to be proven or constructed.

1*. *Sumperasma*: General restatement of the theorem, in terms of situation and result.

Figure 10.26. The RC structure of the general template of a Greek proof. The middle term is the proof (5); the construction (4) remains outside the structure.

2. *Use of X/RC is prevalent in late Classical and Hellentistic culture.* Fourth-century BCE poetry abounds in intricately constructed X/RC forms. (We find an example of this exact, three-step framing RC structure of an inner core in Theocritus's twenty-third Idyll, also known as the *Suicide Paraclausithyron* or *Erastes* (figure 10.27)). In one of the very few long discussions of X/RC in the context of rhetoric (Worthington 1991, 1992),[51] these forms are seen essentially as stylistic calligraphy, or a way of giving more elegant shape to an argument in the stage of "publication" of a speech, mostly for the use of students of rhetoric, through an accepted literary styling device of the time. This is not so in the case of X/RC in mathematics, however: as we shall see below, X/RC patterns go much deeper into the structure of the theorems for them to be mere embellishment.

3. *The most important basic structures at the atomic level of proof are X/RC.* We saw in section 2.2 the two types of links, from atomic sentence to atomic sentence, in narrative: the simpler ones, which are linear and strictly ordered (as in the temporal "the king died and then the queen died") and the more complex ones, of causality ("the king died and then the queen died of grief"), where more than one action or state-of-affairs statement causes another, resulting in a nonlinear pattern (see figure 10.8). Exactly the same is true in the linking of sentences in geometric proof, as we also saw earlier in this section. The links can be either (1) *linear*, or reversible, as in the case of the steps in a construction or the rewrite-rules of the proof, that is, the equivalents of the substitution, say, of $(a + b)^2$ by $a^2+b^2 + 2ab$, or (2) *nonlinear*, or non-reversible, as in the case of any type of structure that involves syllogisms or other more complex structures of thinking. We shall examine some examples of such structures below. Let me just say here that the basic syllogisms of logic that underlie so much of mathematical thinking—though often present in the texts of theorems, as we saw, not in full but in enthymematic form—are in atomic RC form. Thus, the *modus ponens* takes the form of an elementary RC, in figure 10.28, with 1 and 1* pivoting on B. As for the *modus tollens*, a similar structure pivots on A in figure 10.29. However, if we further break down 1 in the basic *tollens* structure in its two constituent propositions ("A is true" and "B is true") and see that the expanded structure is chiastic, the outer pair pivoting on A and the inner on B.

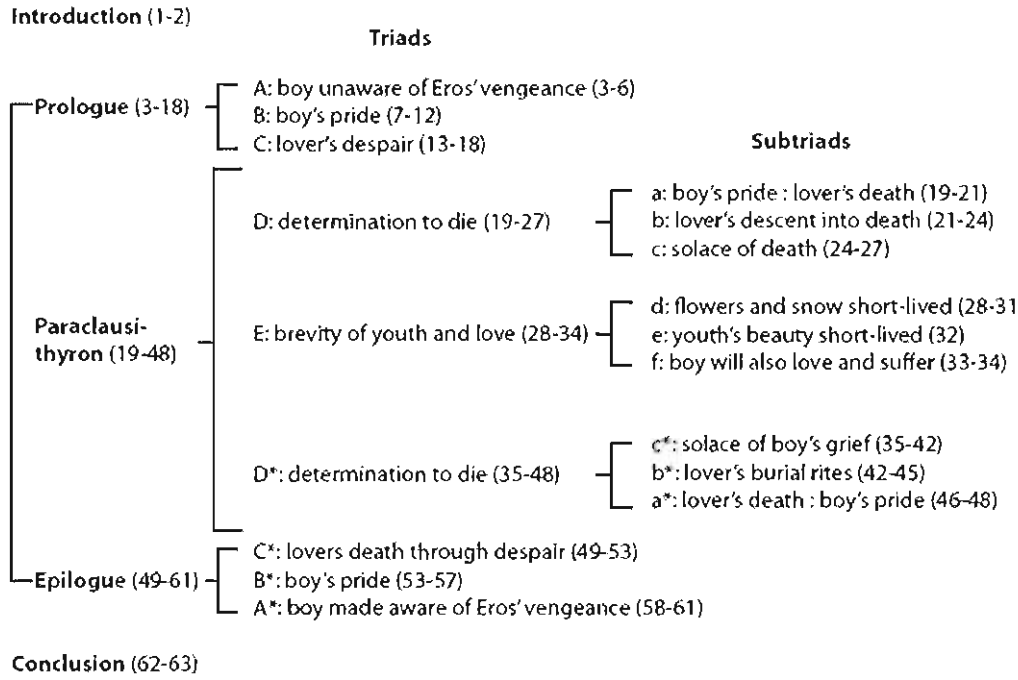

Figure 10.27. The RC in Theocritus' *Suicide Paraclausithyron of Erastes*, as identified by Frank Copley. (Diagram from Griffin 2008).

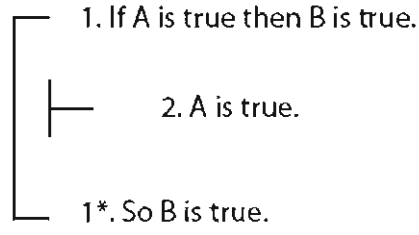

Figure 10.28. The RC structure of *modus ponens*.

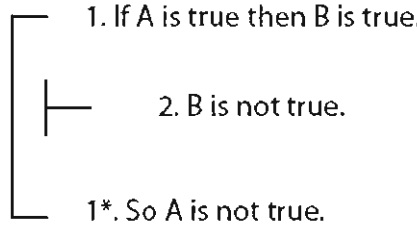

Figure 10.29. The underlying RC structure of *modus tollens*.

4. *The style of Euclidean mathematics provokes in a modern reader the reactions typical of perceiving X/RC-dominated structures.*[52] The anthropologist Mary Douglas has done more than anyone to show the importance of ring-composition outside the confines of philology, in other words as a way of thought, an archaic cultural habit, and a cognitive device. One of the points she makes repeatedly in her important book, *Thinking in Circles*, is that "repudiations of structure [by literary scholars, in a poem or other literary work] are like signposts saying: 'Here lies a hidden ring-composition' " (2007, 101). Accustomed as we are to observing structures with linear, "organic" form, we tend to see texts constructed on principles of X/RC as merely formless or messy. Douglas's argument provides a surprising impetus to look for X/RC in Euclid and other texts of Greek mathematics: simply that their style seems so strange to us, and above all, often so needlessly tedious and repetitive. For senseless repetitions are often a signpost of underlying X/RC.

5.5. Toward a Typology of X/RC in Greek Mathematics

Most of the arguments developed earlier for the usefulness of X/RC in rhetoric also apply, occasionally with slight adaptations, to its use in geometric proof. Similar needs lead to similar forms: what the compass-literate geometric practitioner needs, to make out of sequences of geometric actions and sets of observations a kind of demonstrative argument, is a new syntax. And this is precisely what he finds in X/RC. As Douglas puts it, ". . . as a kind of syntax, the ring form brings ambiguity under control and reduces confusion." (I add that this is equally true of the chiastic form.) The constant flow of individual statements that arise in a geometric investigation can be simply, i.e., serially ordered (the model of action representation borrowed from simple, paratactic temporal narrative) only in the case of constructions. X/RC provides a new way to compose statements in other parts of the theorem, especially the proof, or *apodeixis* one that is much used by Greek mathematicians. The main reason for this is cognitive and not merely aesthetic.

Rather than attempt a general theory of X/RC in Greek mathematics, we shall examine its operation in several theorems, both at the level of microstructure, in the atomic Xs or RCs that appear in the proofs, and at the level of middle or mesostructure, in the forms in which atomic X/RCs are put together to achieve demonstration, completing the *apodeixis* of a theorem. I should warn the reader: my examples here are culled from a few theorems of Euclid, and thus the typology of the uses of X/RC I present does not necessarily exhaust all of its forms.[53]

5.5.1. Microstructure

Atomic sentences are joined together in the microstructure of a proof. When their linking is not linear, not simply in the form of a rewrite rule, the sentences more often than not form X/RCs belonging to one of three types: the *rule-centered RC*, the *binary X*, and the *substitution RC*. These are in essence the atomic forms of mathematical demonstration, roughly the equivalent of elementary syllogisms in logic. They can be combined, concatenated, or interlocked to form longer mesostructures, as we shall see in the next section. I discuss the first two of these types below and, after a necessary detour into the determining characteristics of mathematical proof more generally, the third.

5.5.1.a. The Rule-Centered RC

One of the most distinctive functions of the order- and form-giving qualities of RC is in its emphasizing, out of a series of elements, a particular one, by putting it at its center (Douglas 2007, 38): "A well-marked turning point is a sign of a well-designed ring-composition" (34). Such well-marked turning points abound in Greek mathematical-argumentative style, as they do in rhetorical. They provide the glue, as it were, through which pairs or larger sets of sentences, usually either parallel or contrasting, acquire cohesion, as satellites of an appointed center of gravity. Figure 10.30 shows such an example from a nonmathematical text, yet one constructed with a high degree of the newly discovered, fifth-century BCE rhetorical sophistication. This RC structure is from book 1 of Thucydides' *History*, written at least a century before Euclid.[54] (Though the verb in sentences 2.1–2, 2.3, 2.5, and 2.6 (here condensing a longer text) is the copula *is*, the sentences are of the action type, in the sense of describing concrete things that actually happened.) However, the central unifying element of the RC, sentence 2.4, is in the form of a *gnomê*, essentially expressing a thought, or general principle of the type that Thucydides was so keen on discovering (Lang 1984). We have seen exactly the same pattern of putting the general law in the middle in rhetoric, as the RC type of rhetorical argument (RC in figure 10.16).

2.1-2 Migration was frequent because the defenseless were expelled by those larger.

2.3 The best land was most subject to change—e.g., in Thessaly, etc.

2.4 Fine land conferred power, but this only attracted *stasis* and invasion.

2.5 E.g., Attika with her poor land remained *stasis*-free and settled.

2.6 She grew larger by attracting powerful émigrés and taking them in.

Figure 10.30. An RC structure from Thucydides *History*, identified by J. R. Ellis (1991), with *gnômé*-like statement is in the middle.

We find this rule-centered pattern again and again in Euclid. Its importance cannot be exaggerated: in fact, we might say that this very structure comes as close as any other to marking the point of passage, from the mode of doing/thinking embodied in a compass-literate practice of geometry to a new one. Its first instance in the *Elements*, in the center of proposition 1.1, is reproduced in figure 10.31. In this case the outer two sentences, 1 and 1*, are, like 2, also of the state-of-affairs type. But, very much as in the example from Thucydides, they refer to concrete cases, with the pair (CA, CB) acting as pivot. And, again as in the Thucydidean example, the central sentence is of *gnômé* type, a general rule. As with its progenitor in rhetoric, this particular use of RC can be seen to descend, not just in form but in function, from the Homeric *paradeigma*-type RC (see figure 10.15; see also Willcock 1964). The central narrative *paradeigma* functions in Homeric RC as a higher, prototypical model, or law, exactly as does the common notion in 1.1, or any other "nonartistic" reference to previously established truths, in other propositions of the *Elements*. In the example of figure 10.15, the RC links present action, which pivots on sentences 1 and 1*, to its mythological justification, 3, through the central pair 2-2*. In the generic case, as we see it in Willcock, this has the form: (1) the present situation is presented as a problem; (2) this is taken as a particular case of a more general, because mythological-thus-prototypical, situation; (3) the prototypical myth is recounted; (2*) the answer given to the problem in the myth is identified; (1*) the same answer is made to apply to the situation in hand. The rule-centered RC in figure 10.31 has the exact same structure and function as the three central elements (2-3-2*) of the Homeric generic form, with the difference that the central object changes, from prototypical narrative *paradeigma* to general, abstract law:

1. Thus, CA and CB are each equal to AB.

2. But things equal to the same thing are also equal to one another.

1*. Thus, CA is also equal to CB.

Figure 10.31. The central RC in proposition 1.1 of Euclid's *Elements*.

(1) A concrete situation is described: "CA and CB are each equal to AB".
(2) A generic rule applying to the situation is stated; this gives, as it were, the possibility of a new twist (in italics) to the situation: "Things equal to the same thing *are also equal to one another*."
(1*) This answer is applied to the concrete situation in (1): "Thus CA and CB are equal to one another."

If we let aside for a moment our practice of avoiding the Whig-historical tendency of interpreting Greek intellectual history as "leading toward logic," we could call the rule-centered RC a precursor of the *modus ponens*.

5.5.1.b. The Binary X

In my discussion of the process of division in rhetoric (see section 3.3.2.d), I proposed the poetic form of priamel as the model for the process of setting up counternarratives as foils. The mathematical equivalent of counternarrative is also apparent at the microstructure of geometric proof. This belongs to the general class of negative arguments, of which the most famous is the reductio ad absurdum. As we saw, the form that priamel-type reasoning takes in rhetoric is the following: (1) the rhetor asks (often implicitly) if the narrative he is supporting has alternatives; (2) (if so) he states them; (3) he assumes the first of these is true; (4) he isolates one of its necessary features; (5) he shows that this feature does not arise in the extended storyworld of the case's narration; (6) he brings the actuality and the contradicting counternarrative next to one another, to make this apparent, thus invalidating the counternarrative; (7) he repeats this process for the next counternarratives, and so on; (8) he returns to the narrative in his narration as the true one, that is, the only sustainable version of reality.[55] Of these, the sequence 3-4-5-6 is an X, the two central elements (4-5) contradicting one another. (In our X/RC notation, these would be written as 4-4*.) This is exactly the form of our X-type 2 rhetorical argument (see section 4.3.2). The same pattern drives the binary X form of microstructure in geometric proofs.

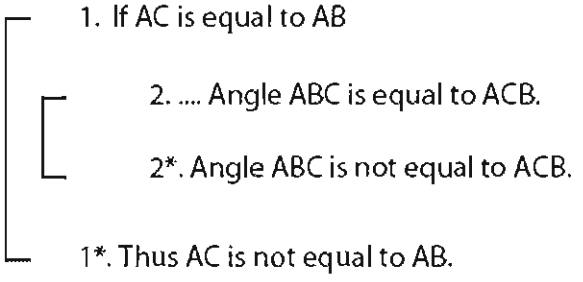

Figure 10.32. An X structure in proposition 1.19 of Euclid's *Elements*.

Here is a very typical example, from the heart of the proof of proposition 1.19 of the *Elements*: "In fact, *AC* is not equal to *AB*. For [if AC is equal to AB] then angle *ABC* would also have been equal to *ACB*. But it is not. Thus *AC* is not equal to *AB*." The second, third, and fourth periods of this sequence form the X shown in figure 10.32. The structural equivalence to the central X of an X-type 2 rhetorical argument is obvious: the outer pair 1-1* pivots on the counterproposition to what the theorem states—as it does on the counternarrative, in figure 10.18. Of the central pair, 2 is a consequence of 1. It is followed by 2*, its negation, which disproves it, as we see in 1*. As we have seen, "saying exactly opposite things" (*enantiotata*) is an indication of falsity at least since Gorgias's time (Spatharas 2001, 398–99).

The statement preceding the first in this example ("in fact *AC* is not equal to *AB*") results from a division in the rhetorical sense: "*AC* is either equal to *AB* or less than it." The binary X then takes the first of these cases ("*AC* is equal to *AB*") and disproves it, or, in other words, proves that it is not true. The full underlying nonlinear form, shown in figure 10.33, would be structurally the same as the one shown in figure 10.20, where solid arrows represent serial order and dotted ones implicational. In the geometric case, "counter-story A" becomes "counter-statement A" (="*AC* is equal to *AB*"), "contains action B" becomes "implies quality B is true" (="angle *ABC* would be equal to *ACB*," imported from proposition 1.5), "action B did not occur" becomes "B is not true" (="angle *ABC* is not equal to *ACB*," as we know from the *protasis*, where it is stated that triangle ABC), and "counter-story A did not happen" becomes

"counter-statement A is not true" ("*AB* is not equal to *AC*"). Needless to say, as a representation of this non-linear form, the sequence of sentences in figure 10.32 is gappy, both in that it does not begin with a clear statement of that which is to be disproven ("*AC* is equal to *AB*"), and in that it leaves step *i* of the diagram in figure 10.33, from 1 to 2—the (unstated) application of proposition 1.5 —unexplained. But this is exactly the same type of gappiness that we find in its equivalent in rhetoric. Thus—as was also said of the rhetorical X-type 2—the power of the binary X in geometric proof is that it linearly encodes a nonlinear pattern.

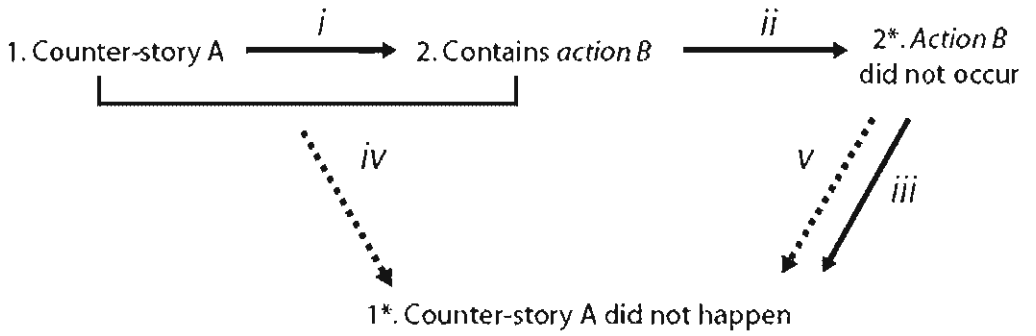

Figure 10.33. A non-linear depiction of the connections of the elements of a rhetorical X-type 2 structure.

5.5.1.c. An Aside: The Levels of Proof

It is useful at this point to look at the question of the degree of completeness—viewing completeness, in this case, as the opposite of gappiness—of Greek proofs, starting with the reminder of the basic point, from the cognitive side: to the extent that a text is a representation of either an external or an internal reality, it is of necessity gappy (see section 2.1). This applies also to mathematical proofs, by virtue of their being texts: regardless of the medium of their encoding, all proofs are gappy, and even more so—as mathematicians of old did not possess our own advanced criteria of logical rigor—Greek proofs.

Always keeping in mind that Greek proofs have the distinctive characteristic of the centrality of lettered diagrams Netz (1999a), we can distinguish roughly four levels of completeness of a Greek mathematical proof, moving from the lower to the higher. (I do not consider here cases of the simple carrying out of arithmetic operations, in either pure arithmetic or calculational geometry, but only processes involving logical deduction.) Though these emerge in historical sequence, that is, level n before level $n + 1$ previously established levels of proof are not discarded by progress but exist on the side of the newer ones. Thus, a mathematician fluent in level n will also deal on occasion in levels $n-1$, $n-2$, and so on. It is a well-known fact that, even to our day, a big part of the process of mathematical discovery occurs at a lower level of completeness than that in which it is finally written for presentation.

Level one, the earliest, is mostly deictic, manual, and visual. This is the level of the practical geometer, the person motivated in the use of geometry by the tasks and problems of painting, drawing, or architecture or even, possibly, early abstract theoretical investigations. Though, like a grandmaster playing blind chess, a highly experienced geometric practitioner can on occasion work purely with internal representations of shapes, level one is mostly externally realized, involving a practitioner's eyes and hands, as well as his brain, as he is creating or merely looking at diagrams.[56] The actions of drawing, gesturing, pointing, and even the occasional scribbling of notes (some of them in diagrammatic language) can also occur, as can some vocal expressions, hums, exclamations ("Eureka!"), and so forth. But what holds the sequence of external actions, diagrams, and symbols together is an internal representation, a mentally encoded sequence of actions. This may occasionally exist without any words at all, somewhat like a silent film or kinesthetic algorithm unfolding, unifying the external actions.

(Paradoxically, this is more easy to conceive of today, than it would have been until a decade ago, as such visual-cinematic geometric proofs are now often seen on mathematical websites.) There need not be an abstract notion of "logical rightness" at this level, merely the conviction that a technique produces the required results, a conviction probably beginning in compass literacy, in other words practical, geometric mathematics, whose main aim is the construction of shapes with certain qualities.

Level two is guided by the need of a geometric practitioner to communicate to a peer or student his belief in the rightness of a geometric construction technique or general assertion. This level is basically oral and performative, while still retaining a strong basis in manual work, such as drawing. But the practitioner has to use at least some verbal language to support his actions ("I draw a line parallel to . . ."), deixis ("*this* is bigger than *that*"), and visually-based arguments ("It's the same, do you see?"), which weld together the sequence of mental representation. As in all live interaction, statements can be contested ("But *why* is *ABC* equal to *EFG*?"), a process also leading to reflexivity, or the internalization of contestation, and the creation of arguments to preempt it, before being asked to do so by a peer.

The need that motivates *level three* is the recording of a proof in a form that can be preserved and transmitted. It is the first level at which a demonstrative argument has to be put in writing, usually on a wax tablet or papyrus. This is the first level for which we have direct historical evidence. We find it in the *arhaikon ethos* (see section 5.3.1), a style that, we can assume from Simplicius's reference to it, was the rule in the early days of deductive mathematics, in the fifth century BCE. Simplicius calls this style *hupomnêmatikon*, i.e., an aid to memory. We can surmise, reading this comment in context, that it is in the form of notes by the mathematician, either to himself or a few peers, containing diagrams, possibly nonlettered, as well as brief comments giving the basic steps of a proof. Level three proof is expert-to-expert: it contains enough evidence to guide a knowledgeable practitioner to reconstruct a proof, but probably not enough to transmit it coherently to a less trained person. The *arhaikon ethos* is the style of the time when the creation and transmission of theorems was limited to a small intellectual coterie.

Level four is the level in which most Greek mathematics is written, the level also of Euclid's *Elements*.[57] This level, which characteristically employs lettered diagrams, is the first level at which the written form of a deductive argument is more or less self-contained, in the context of the more general work in which it appears. In other words, given the information at level four even a nonexpert reader reconstruct the arguments of a proof, without resorting to the physical use of ruler and compass. Though in the last two centuries there has been much criticism of the rigor of Euclidean proofs, almost all of them remain *good enough* even by modern standards, in the very solid sense of being able to convince a professional mathematician of the truth of the statements they are meant to prove. The reason they are considered the first, historically, to merit the name of proofs is not that they are in any sense "perfect" but that they give enough material for a mathematician to be convinced a proposition is true, and can be reworked to reach the degree of completeness that his or her sense of rigor demands. Level four proofs can also be thought of as a kind of shorthand for a higher, more complete proof, though they are not themselves, technically speaking, complete. In fact, we could say—paraphrasing Heraclitus on the mantic Apollo—that Euclidean proofs neither speak nor hide their meaning but signal it.[58]

The practice of looking closely at the actual texts of Greek proofs, as they were written, instead of examining the underlying mathematical arguments, has been a crucial factor in the revival of the study of Greek mathematics over the past few decades. However, we must not go to the other extreme of actually forgetting that such arguments exist: after all, all texts are gappy, and the people who wrote the early proofs knew this as well as we do. (If we need proof of this knowledge, there is nothing stronger than the word *hupomnêmatikos*, characterizing *arhaikon ethos*, the early proof style: to be reminders, proofs had to be reminders *of* something, that is, a more complete process.) To do full justice to Greek proofs we must always remember the cognitive grounding of any mathematical proof, Greek proofs included, i.e., the fact that it is

ultimately justified by a mental processes of comprehension, and this involves, among other things being aware of yet a *fifth level* of proof. The fact that we do not have full examples of it in the extant texts of Greek mathematics should not blind us to the fact that the Greek mathematicians were aware of its existence, at least as a potentiality.

More specifically, we saw earlier (in section 5.3.1.a) that the internal comparison of various propositions of the *Elements* shows us that Euclid can be more or less complete in his proofs. Though as a rule he works enthymematically, omitting obvious points, he occasionally tends toward an altogether higher level of completeness than is typical of level four. What is more, Euclid's inconsistencies in style are not random but are usually driven by a rationale of reader-friendliness: he gives more information when this is helpful, less when more becomes tedious. None of this comes as a surprise to modern mathematicians. The degree of detail required in a written proof, even today, is determined by factors that include its intended audience, the nature of the occasion or publication in which it is presented, as well as—to a degree—the personal preferences, styles, and habits of the writer. However, even the proof that mathematicians speak of as *rigorous*, or complete enough for it to be accepted for publication as an original contribution in a peer-reviewed journal, is far from complete from the logical viewpoint. Nicolas G. de Bruijn has calculated that a proof written out in the completeness required to satisfy the strict standards of a computer proof-checking program is roughly 3.7 times the length of the same proof in its peer-acceptable rigorous form (1980; Wiedjik 2000.) Leaving computers aside, we can view de Bruijn's standards as describing the level of completeness in the write-up of a proof that would satisfy even the most complete logical pedant or, alternatively, as that level of completeness from which all answers to questions of clarification or puzzlement at the written exposition of a proof can be culled.

Thus, we can profitably see level four proofs in Greek mathematics as resulting from a tension between level three proofs, those in the *hupomnêmatikon êthos*, the unrealized, ideal level of thoroughness, which I call *level five*. Generalizing from the earlier discussion of the explicit mention or nonmention of the use of common notions (in section 5.3.1), we can say that a level five proof of a proposition in the *Elements* would contain, in addition to (1) any intermediate steps in an argument, which Euclid often omits as obvious; (2) reference, at the appropriate point, to every single one of the definitions, common notions, axioms, and previously proved theorems (the "nonartistic" proofs) used in the proof; and (3) the digressions taking us from the main proof to each and every nonartistic one, usually in the form of the rule-centered RC, to show how it is applied. Thankfully, the pedantry of level five is never fully realized. But (also thankfully) its invisible pressure is a great contribution to the form of Euclidean proofs.

5.5.1.d. The Substitution RC

The discussion of levels of proof is especially useful in understanding the form of the third kind of X/RC used at the microlevel of Greek proofs, the *substitution RC*. Unlike the rule-centered RC and the binary X, I have not found an exact equivalent of the substitution RC in Greek rhetoric. This makes sense, as this is an elaborate, delicate construction, and for it to work there are requirements not provided by the cultural context of rhetoric: a more sophisticated audience to decode it, the certainty-preserving operations of mathematics and the clarity and precision of written expression.

To understand the substitution RC, we have to go back to Aristotle's (pre-Euclidean) theory of the tripartite syllogism, with the associated concepts of the *middle term* (*meson*) and the two *extremes* (*akra*). Let us consider the most famous example of syllogism:[59]

1. Socrates is a man.
2. All men are mortal.

3. Socrates is mortal.

(1) and (2) are here the *premises*, and (3) the *conclusion*. The *middle term* is defined as the shared element of the two premises—in this example, "man"—and the two *extremes* as the two terms not shared, here "Socrates" and "mortal." By Aristotle's theory, the conclusion (3) connects the two extremes, an effect made possible by the existence of the middle term.

We can write the previous triad in generic form as:

1. A has property a.
2. All x's with property a have property b.
3. A has property b.

This shows it to be structurally equivalent to the composition in figure 10.31. Though one rarely meets constructions of this formal completeness in the written form of the proofs of Euclidean mathematics, they are definitely there in the level five arguments on which the written proofs rely, and in the example of figure 10.31 they are also in the actual proof.

Let us now turn to *Elements* 1.7, whose proof is typical of level four. Below is the theorem in its full form with its diagram, figure 10.34, with words in parentheses added for clarity and those in brackets indicating previously established (nonartistic) truths which are made use of in the proof:

> On the same straight-line, two other straight lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at different points on the same side (of the straight-line), but having the same ends as the given straight-lines.

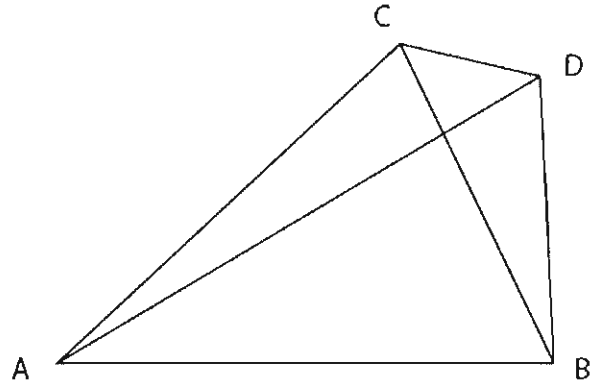

Figure 10.34. The diagram accompanying the proof of proposition 1.7 of Euclid's *Elements*.

> For, if possible, let the two straight-lines AD, DB, equal to two given straight-lines AC, CB, respectively, have been constructed on the same straight-line AB, meeting at different points, C and D, on the same side (of AB), and having the same ends (on AB). So CA and DA are equal, having the same ends at A, and CB and DB are equal, having the same ends at B. And let CD have been joined [Axiom1].
>
> Therefore, since AC is equal to AD, the angle ACD is also equal to angle ADC. [Proposition 1.5]. Thus, ADC (is) greater than DCB [Common Notion 5]. Thus, CDB is much greater than DCB [Common Notion 5]. Again, since CB is equal to DB, the angle CDB is also equal to angle DCB [Proposition 1.5]. But it was shown that the former (angle) is also much greater (than the latter). The very thing is impossible.
>
> Thus, on the same straight-line, two other straight-lines equal, respectively, to two (given)

straight-lines (which meet) cannot be constructed (meeting) at different points on the same side (of the straight-line), but having the same ends as the given straight-lines. (Which is) the very thing it was required to show.

The main difficulty of the proof part of the theorem (starting with "Therefore, since AC is equal to AD, the angle ACD is also equal to angle ADC") is to prove that *CDB* is greater than *DCB*. There are only three statements in this paragraph, creating a very grappy whole:

1. Therefore, since *AC* is equal to *AD*, the angle *ACD* is also equal to angle *ADC*.
2. Thus, *ADC* (is) greater than *DCB*.
3. Thus, *CDB* is much greater than *DCB*.

Clearly, here we are not dealing with a syllogism in the Aristotelian sense. However, there is a formal analogy with *meson*, in the term *ADC*, which, like the *meson*, is mentioned in both the first and second statement. In a syllogism, the third statement, the conclusion, should be connecting *ACD* and *DCB*, the two extremes (*akra*) by this analogy. But though *DCB* is mentioned in the final statement, *ACD* is absent, and *CDB* has taken its place. Where did it come from, in this sequence of statements? To see this, we must fill the gaps of the argument, through recourse to level five proof. In a more complete version of the proof, the statements written in italics, 1a, 1b, and 2a, would also be required:

1. Therefore, since *AC* is equal to *AD*, the angle *ACD* is also equal to angle *ADC*. [Proposition 1.5]
1a. *Now ACD is greater than DCB.*
1b. *But ACD is equal to ADC.*
2. Thus, *ADC* (is) greater than *DCB*. [Common Notion 5]
2a. *But ADC is part of CDB.*
3. Thus, CDB is much greater than *DCB*. [Common Notion 5]

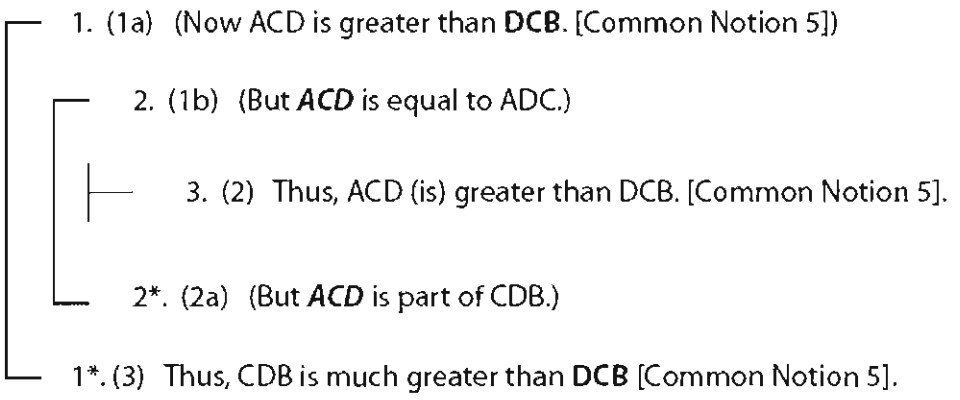

Figure 10.35. An RC structure of statements in proposition 1.7 of Euclid's *Elements*. The new numbering of the statements stresses the RC; the previous is in parentheses.

Looking at sequences 1a-1b-2 and 2-2a-3, we see that each one is a perfect tripartite syllogism, including middle term and extremes: in the first, the middle term is *ACD* and the extremes, connected in 2, *ADC* and *DCB*. In the second, the middle term is *ADC*, and the extremes, connected in 3, *CDB* and *DCB*. If we now merge these two sequences, by using 2 only once, the original statement 1 is developed in a perfect, highly symmetric fivefold RC shown in the diagram in figure 10.35. (I have kept the previous numbering in parentheses.) 1-1* pivots on *DCB*, 2-2* on *ACD*. Another way to describe this extremely powerful formal tool, which I call the *substitution RC*, is the way of merging two syllogisms into one, effected by a substitution

(1b). This is a strong case of the use of RC for giving order, and keeping track, of a complex sequence of statements, in a concrete form that follows the flow of an argument in a continuous way, from 1 to 1*. In fact, it is the RC that represents the full (level five) structure of the argument.

5.5.2. Mesostructure

The three atomic X/RC forms we examined can be put together in sequences to construct longer demonstrative arguments. These sequences are constructed in ways that also, wholly or partly, rely on X/RCs. I have found five types of these longer mesostructures—clearly, the examination of a bigger range of Greek mathematics may reveal others. These are the *long RC*, the *string RC*, the *priamel proof*, the *exhaustion proof*, and the *complex* or *tiered RC*. These forms of structures can also be combined recursively: a string RC can contain a long RC and a string RC, or a complex RC can consist, at one tier, of long RCs, and so on. But this recursiveness, when it occurs, does not as a rule go beyond three steps.

5.5.2.a. The Long RC

This is the most aesthetically pleasing form as it extends a central, rule-centered RC form, to subsume the whole theorem; in this case, the only part of the theorem that is excluded from the RC structure is the construction. Proposition 1.1 of the *Elements* is a good example of an overall RC structure. This is a proposition of the construction type ending with "which is the thing it was required to construct [*poiêsai*]." Connecting this with previous geometric practice, one might see it as putting the practical observations of the maker of the amphora shown in figures 10.22–10.24, i.e., compass-literate vase painter of the late Archaic and early Classical Age, in structured form, with a beginning (a task), a middle (a method, justified) and an end (the task achieved). In this *long RC* form (figure 10.36), the central RC, 5-6-5*, is a rule-centered RC (see section 5.5.1.1). This RC, as in most Greek proofs, is constructed using repetitions, either of whole sentences (1* repeats 1) or parts thereof. Thus, 2-2* pivots on the "given finite straight line," 3-3* on "the triangle ABC," 4-4* on "*CA*, *AB*, and *BC*" and 5-5* on the pair *CA* and *CB*.

1. To construct an equilateral triangle on a given finite straight-line.

2-3. Let AB be the given finite straight-line. So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Axiom 3], and again let the circle ACE with center B and radius BA have been drawn [Axiom 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another to the points A and B (respectively) [Axiom 1].

4. And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB.

5. Thus, CA and CB are each equal to AB.

6. But things equal to the same thing are also equal to one another [C.N. 1].

5*. Thus, CA is also equal to CB.

4*. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

3*-2*. Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB.

1*. (Which is) the very thing it was required to do.

Figure 10.36. The full RC form of proposition 1.1 of Euclid's *Elements*.

The formal beauty of proposition 1.1, as a perfect long RC, derives from the fact that it involves a single argument, a simplicity we rarely find in more complex propositions.

1. To place a straight-line equal to a given straight-line at a given point (as an extremity).

2. Let A be the given point, and BC the given straight-line. So it is required to place a straight-line at point A equal to the given straight-line BC.

For let the straight-line AB have been joined from point A to point B [Axiom 1], and let the equilateral triangle DAB have been constructed upon it [Prop. 1.1]. And let the straight-lines AE and BF have been extended in a straight-line with DA and DB (respectively) [Axiom 2]. And let the circle CGH with center B and radius BC have been drawn [Axiom 3], and again let the circle GKL with center D and radius DG have been drawn.

3. Therefore, since the point B is the center of (the circle) CGH, BC is equal to BG [Def. 1.15].

4. Again, since the point D is the center of the circle GKL, DL is equal to DG [Def. 1. 15].

5. For then angle ABC would also have been equal to ACB.

4*. Thus, the remainder AL is equal to the remainder BG [C.N. 3].

3*. But BC was also shown (to be) equal to BG.

6. AL and BC are each equal to BG.

7. But things equal to the same thing are also equal to one another [C.N. 1].

6*. Thus, AC is not less than AB.

2*. Thus, the straight-line AL, equal to the given straight line BC, has been placed at the given point A.

1*. (Which is) the very thing it was required to do.

Figure 10.37. The string RC form of proposition 1.2 of Euclid's *Elements*.

5.5.2.b. The String X/RC

In this form, independent X/RCs are concatenated, each incrementally forming a step toward the proof. The proof part of *Elements* 1.2 is a good example of this type of structure. Again, we have a proposition of the construction type (figure 10.37). The first part of the string is the long RC, 3-4-5-4*-3*. Here, 3-3* pivots on the equality of *BC* and *BG*. For the pivot of 4-4* to become apparent, it helps to see the accompanying diagram, shown in Figure 10.38, as the logic of demonstration proceeds—as so much in early Greek mathematics—from the image: the pivots of 4-4* are the two lines *DL* and *DG*, which are taken whole in 4, but with the fragments *DA* and DB subtracted, respectively, in 4*.

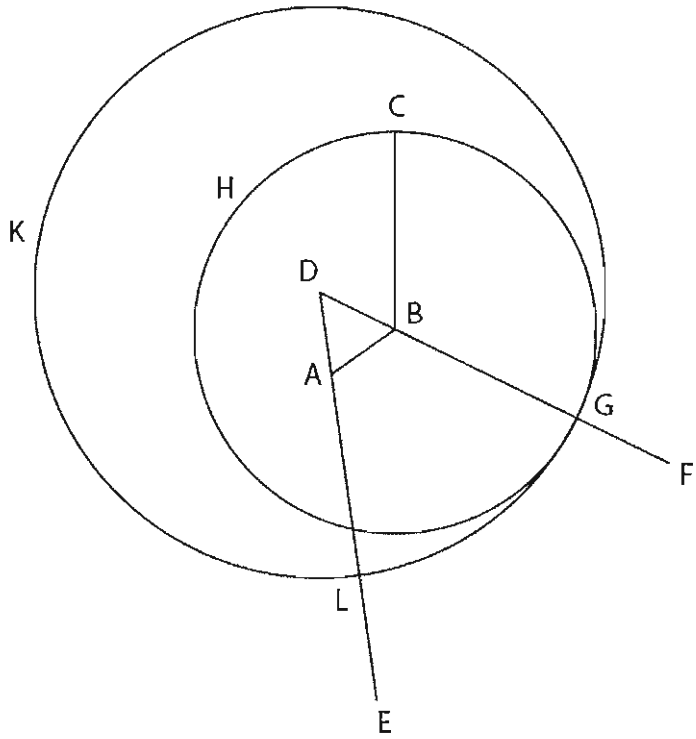

Figure 10.38. The diagram accompanying the proof of proposition 1.2 of Euclid's *Elements*.

The second part of the string, is a rule-centered RC, 6-7-6*, exactly of the type we saw in proposition 1.1.

5.5.2.c. The Priamel Proof

In section 3.2.2.d, we saw how the process of division operates in early Greek rhetorical proofs. The poetic form of priamel was proposed as model for these. The rationale of division also applies to some mesostructures in mathematical proofs, with the priamel-type construction employed, as it is in poetry, as "a focusing or selecting device *in which one or more terms serve as foils* for the point of particular interest" (Bundy 1962; my italics). In rhetoric, the foils are presented in the form of counternarratives to be discarded as less likely than "the point of particular interest," that is, the speaker's own narration; in mathematics, a similar process of priamel-type division operates in which "counternarratives" take the form of *counterstatements*, that is, statements that counter that of the theorem. Because of the differences of rhetorical and mathematical argument (see section 5.4.1), "less likely" (in rhetoric) becomes "not true" (in mathematics). Priamel-type thinking is the basis of many Greek proofs. By this, a theorem's *protasis* proposes condition A to hold; the counterstatements to it, assuming, in turn, conditions B and C, are stated at the start of the *apodeixis*; the rest of the *apodeixis* then proceeds by disproving B and C, thus leaving A as the only answer.[60]

1. In any triangle, the greater angle is subtended by the greater side.
    2. Let ABC be a triangle having the angle ABC greater than BCA, I say that AC is also greater than side AB.
        3. For if not, AC is certainly either equal to, or less than, AB.
            4. In fact, AC is not equal to AB.
                5. For then angle ABC would also have been equal to ACB.
                5*. But it is not.
            4*. Thus, AC is not equal to AB.
            6. Neither, indeed, is AC less than AB.
                7. For then angle ABC would also have been less than ACB.
                7*. But it is not.
            6*. Thus, AC is not less than AB.
        3*. But is was shown that AC is not equal to AB either.
    2*. Thus, AC is greater than AB.
1*. Thus, in any triangle, the greater angle is subtended by the greater side. (Being) what it was required to prove.

Figure 10.39. The priamel X/RC structure of proposition 1.19 of Euclid's *Elements*.

For a good example of the priamel proof, let us turn to *Elements* 1.19, a part of which we saw earlier in section 5.1.1.2 as an example of binary X (see figure 10.32). The full proof is shown in figure 10.39. The outer couples of the structure, 1-2-3-3*-2*-1*, frame the two inner binary Xs, 4-5-5*-4* and 6-7-7*-6*, determined by statement 3, the basic priamel-type division: "For if (it is) not (greater), *AC* is certainly either equal to, or less than, *AB*." In a level five proof, both binary Xs included here would be expanded, the first to incorporate a rule-centered RC referring to proposition 1.5, the second to proposition 1.18.[61]

5.5.2.d. The Exhaustion Proof

The exhaustion proof is also driven by a process analogous to rhetorical division of the priamel type, where the analogy is not to counternarratives but to subnarratives, that is, to more detailed *substatements* of the main statement of a theorem. In other words, the thing to be proved is here given more detail, not in order to discard false alternatives in favor of a true one but, as in the analogous case in rhetoric, to reduce a larger problem to manageable parts that can be then attacked piecemeal.

A case of this is already present in what is probably the earliest proof extant, Hippocrates of Chios's squaring of the lunes (ca. 430 BCE). Assuming, as we have grounds to, that the form of the proof we possess is approximately close to its original,[62] this makes its structural similarities with the priamel-type arguments of Gorgias and Antiphon particularly striking—for these are publicly spoken in the same decade as Hippocrates proves his theorem.[63] The strategy of subnarratives in Gorgias's *Encomium of Helen* (see section 3.2.2.d) is exactly the same as the one employed by Hippocrates in his proof. The theorem presents a method for squaring a particular class of lunes, whose boundaries are arcs of circles obeying certain constraints. The proof proceeds in three steps, with Hippocrates treating the three different possibilities for the outer boundary separately, thus breaking the statement into three substatements: it can be exactly a semicircle; it can be larger than a semicircle; it can be smaller than a semicircle. As also in rhetoric, the division in a mathematical proof may either be set out whole, in a special section of the theorem, or piecemeal, each of its elements before the part of the proof that applies to it.

An example of the former case in rhetoric can be seen in the *Encomium* and, in mathematics, in *Elements* 1.19 (statement 3 in figure 10.39); as for piecemeal division, we can see it in Antiphon's *First Tetralogy* (rhetoric) and in Hippocrates' quadrature of the lunes (mathematics).[64]

5.5.2.e. The Complex or Tiered X/RC

Complex or tiered X/RCs are treelike structures in which elements of X/RCs branch into further ones of the same or different type. Thus, an element of an X can branch off into another X *or* an RC, and an element of an RC can branch off into an X *or* an RC. These can then break down more, into further X/RCs. Below, I discuss evidence for the frequent occurrence of a two-tiered RC in the level five version of many proofs, which is more usually taken for granted—though sporadically also explicitly referred to—in recorded level four proofs. Though I have not found evidence of more tiers in the theorems I have examined, I think it is probable that they exist in other Greek theorems.

The first examples of complex X/RC are to be found in poetry. The structure is explicitly there already in Homer and lyric poetry but not, interestingly, in fifth- and early fourth-century rhetoric. Though simple X/RCs of the types we examined in section 4 abound in the works of Gorgias, Antiphon, and Lysias, in these there are no complex X/RCs. The first evidence of such complex forms in prose is in the *History* of Thucydides (Ellis 1991), and this mostly in the parts where historical evidence is scant or altogether absent—in other words, complex X/RC in prose smacks of fabrication or, more kindly, "stuffing." This is also the opinion of Worthington (1992), for its use in rhetoric.[65] He writes: "I do not say the device *in itself* [author's italics] is a sign of written revision, since the oral version of the speech probably used ring-composition, that is repetition, [*sic*] most obviously as an aide to memory of the speaker and for repeating arguments and appeals in order to counteract wandering attention on the part of the listeners. In composition of the oral speech, where the audience was an aural and doubtless inattentive one, may have been something akin to the primary level, with no subdivisions of rings within rings [i.e., complex RC]." (1991, 62).[66]

One of the cognitive functions of X/RC could well be its use as a memory aid (Thalmann 1984). With this in mind, it is worth discussing the possible spatial analogues of X/RC (a discussion, incidentally, that I have not found in the existing literature), analogues that can explain both the usefulness and usability, even in a purely oral context, of complex X/RCs.[67] It is known that Greek orators used the memory device of *technê mnêmonikê*, whose invention is traditionally attributed to Simonides of Keos.[68] In this method, the material to be memorized is associated with particular places in the mental representation of a course through a well-known space, either private (e.g., the orator's house) or, more often, public (e.g., a course from one part of town to the *agora*), with items to be memorized mentally "stored" in particular places.[69] The technique is ideally suited for the imposition on the memorized material of *ordering*, *modularity*, and *tiering*. The first is achieved through the order that mentally "walking" the course naturally imposes: item *a* comes before *b* in the memorized material if and only if the place associated with it is reached, in the imagined physical course, before that of *b*. Modularity is likewise imposed by the corresponding modularity of space: thus, if the space in the memory plan is a house with many rooms, each room can correspond to a separate part of the speech, which naturally groups together the elements of the memorized material placed "inside" the room. A similar equivalence applies to tiers: a room in the house that functions as a memory plan can be listed as one item, at level *n* of organization, but one can also "enter" the room, where a finer level of organization, $n+1$, can be found, if extra detail is needed. The item, which describes the room at level n, is thus a "chapter heading," whereas the items inside the room, in the $n + 1$ level reading, are the contents of the chapter. (3, 6, and 9 in figure 10.40 can function

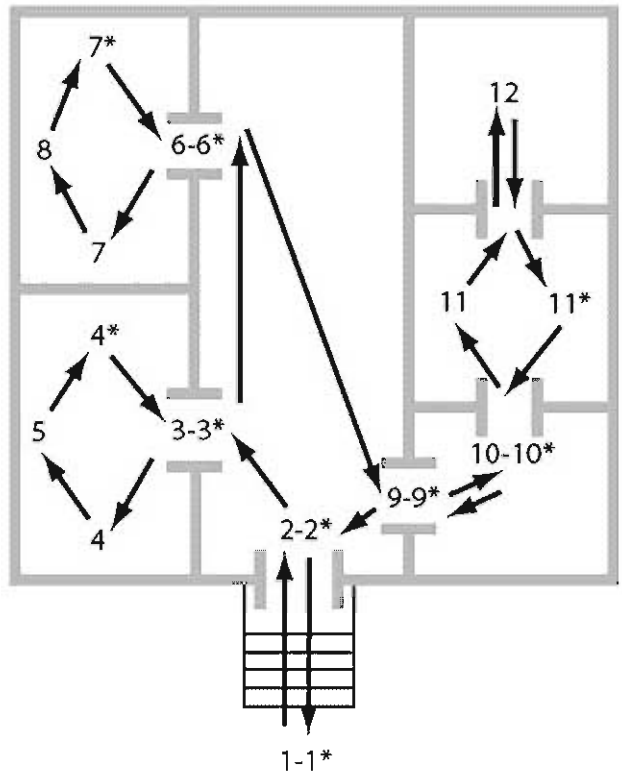

Figure 10.40. A mental route through a building, as a basis for a two-tiered X/RC structure.

If, now, during the mental course through the chosen memory plan an orator retraces his steps, that is, *moves in a certain direction*, then *stops at a particular point*, and then *turns back* and retraces his steps, the items associated with the places he passes, when written down in sequence, form an RC. What's more, if the course includes such going-to-and-coming-back routines as *parts*, the resulting RC comes out as complex, or tiered. Figure 10.40 shows an example of such a course with two tiers, using as its memory plan the layout of a house. The course is indicated by the arrows. The speaker thus "enters" the memory plan at 1 and follows the course indicated by the arrows, assigning one item of the memory list at each place indicated by a digit. In this sense, one has "chapter headings" 3, 6, and 9 at the doors of the respective rooms. A one-tiered description of this linear course from door to door could be framed by the two outer rings 1-1* and 2-2*. The resulting structure would be as shown in figure 10.41, where 3, 6, and 9 are in simple sequence. However, if one follows the full course, also *entering* the rooms, we get a two-tiered RC, which we can diagram as in figure 10.42.

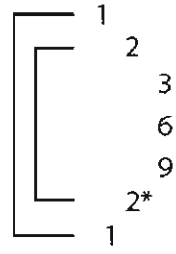

Figure 10.41. The first tier of the X/RC structure resulting from the route in figure 10.40.

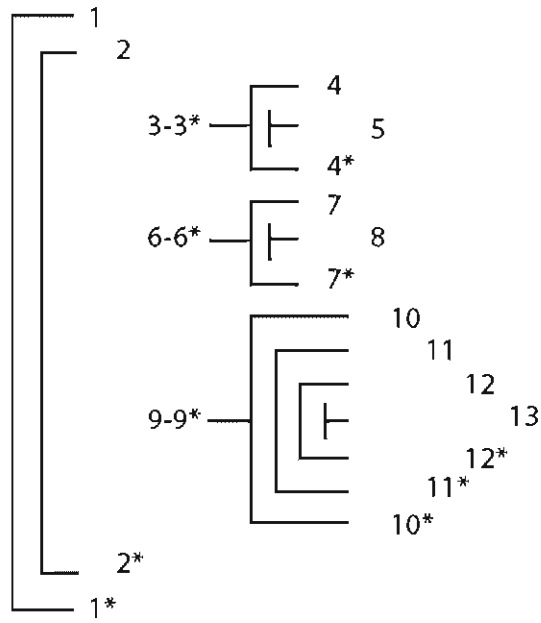

Figure 10.42. The full two-tier X/RC structure resulting from the route in figure 10.40.

Tiering enters the discussion of even the simplest Greek proofs, either at level three or at level four, but always keeping in mind the (at least partly implicit) level five proofs into which, more often than not, further branching occurs. In this sense, the unspoken arguments in Greek proofs often give the strongest indications of the importance of recursiveness, a habit whose origins could well lie in the overlap of spatial mnemotechnics and familiarity with RC in poetic forms: because RC is a perfect way to depict complex spatial courses of the type shown in figure 10.40, it can easily support the habit of going into smaller or greater detail, as the need may arise.[70]

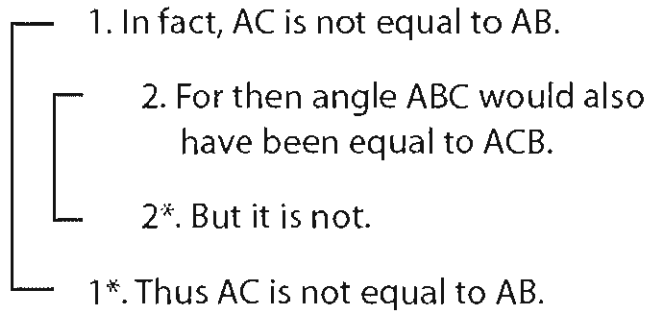

Figure 10.43. A binary X structure in proposition 1.19 of Euclid's *Elements*.

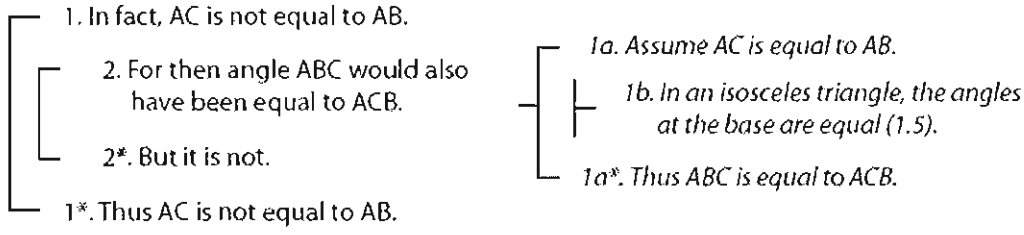

Figure 10.44. The binary X structure in figure 10.43, with the added tier, of the rule-centered RC referring to proposition 1.5.

That the silent arguments behind the written word are implied in Greek proofs, and, what is more important, that their writers *know* that they are, can be safely surmised from the significant

fact of the exceptions, mentioned earlier (see section 5.3.1.a), in which the silent is actually spoken. We already saw the example of the explicit statement, or not, of common notion 1 in the propositions of book 1 of the *Elements*. When it is mentioned, the reference to the nonartistic proof of the common notion takes the form of a simple, basic RC, like the 5-6-5* sequence in figure 10.36. Let us look more closely at a more complex case, involving an implied reference, to see how it branches out, as an RC, from an X structure. This is our earlier example from proposition 1.19 (figure 10.39), where (2) assimilates the conclusion of the application of proposition 1.5—in this function, the equivalent of (5*) in figure 10.36. At level five, the binary X-structure in figure 10.43 would contain, in expanded form, the rule-centered RC centering on 1.5, as shown in figure 10.44. In other words, the binary X-structure subsumes only the conclusion 2, as shorthand for the whole RC 1-1b-1a*.

The idea of truths-within-truths, which may be either explicitly presented or not, is the basis of the recursive nature of Greek, and thus also modern mathematics[71]: new theorems build on older ones, either contained in earlier parts of the longer work containing the theorem or even in other works. They are mentioned if the theorem prover deems it necessary, but not if he or she considers the connection obvious. Here is an example of this exact same Greek process from one of the most famous proofs of modern mathematics, Andrew Wiles's (1995, 516) proof of Fermat's last theorem:

> To change from an $\mathcal{O}$-algebra to an $\mathcal{O}_1$-algebra is straightforward (the complete intersection property can be checked using [Kul. Cor. 2.8 on p. 209]), and to change from Se to ord we use (1.4) and (2.32). The change from str to ord reduces to this since by Proposition 1.1 strict deformation and Selmer deformation sre the same. Note that for the ord case if R is a local Noetherian ring and $F \in R$ is not a unit and not a zero divisor, then R is a complete intersection if and only if R/f is (cf. [BH, Th. 2.3.4]). This completes the proof of the theorem.

Here we see the proof developing, in this sense, exactly as in Hippocrates of Chios's squaring of the lunes, making use of previous results by other mathematicians, for example the references to "Kul" and "BH," or, as in Euclid, making use of earlier results in the same work, such as "(1.4)" and "(2.32)." A certain step in the proof is omitted as "straightforward," and certain concepts such as Selmer deformations or local Noetherian rings are mentioned without further comment, as any expert reader—these are the only readers that would be reading this paper with a chance of understanding it—are supposed to know them. Should one want to include any of these in the proof, an extra layer of detail would appear, in its full form (most of which is suppressed, to avoid pedantic overexplanation) forming locally a small RC—the repetition of the statement of (1.4), say—at the beginning and end of its proof.[72]

5.6. The Final Argument: RC for the Sake of RC

I have given evidence that X/RC is extensively used in early Greek mathematics, both in the macro-, meso-, and microstructure of proofs.

If it were only there in the macrostructure—not just in one outer pair, as Netz (2009, 124) remarks,[73] but in three (see section 5.4.2 and figure 10.26)—it might possibly represent a later imposition of order; if it only existed at the atomic level of rule-centered RC (see section 5.5.1.1) and binary X (see section 5.5.1.2), one might think it was unpremeditated, or even wrongly viewed as X/RC, being merely a case of simple repetition. But the fact that it is *almost everywhere*, in other words at macro-, meso- *and* microlevels, from the triple outer framing of almost every theorem to the long overarching RCs (as in *Elements* 1.1), to the ubiquitous X/RC-based mesostructures, to the complexity of the substitution RCs, to the microstructures

we identified organizing the atomic level of statement arrangements—all this must make us think twice before thinking of X/RC as either intellectual calligraphy imposed on preexisting proofs or an illusion, a pattern that is not really there, a mirage seen by us, as observers.[74]

I shall offer one further argument for the importance of X/RC in early Greek mathematics, an argument that seems hard to dismiss as calligraphy or illusion. This is essentially an argument for the fact that X/RC was a *consciously applied technique* and, moreover, that the first mathematicians saw X/RC as a sine qua non of proof, one of its most essential tools.[75] This is in an example from Euclid's *Dedomena*, or *Data*, as the book's title is usually translated into English,[76] a book some scholars find enigmatic in both authorship and intention.[77] One of the theories about the *Data* is that it is—even more so than the *Elements*—a compilation of earlier mathematicians' theorems. As Taisbak puts it, "perhaps Euclid of Alexandria did nothing to the work but collect and copy the theorems from older sources" (2003, 8). But even if this is so, it makes the *Data* especially important, as a source of earlier mathematics.

One of the more puzzling features about the *Data* is the style of its proofs, a style even more markedly explicit, to the point of being outright pedantic, than that of the *Elements*. (This difference has been attributed by some to the fact that the proofs in the *Data* rely more on the technique of geometric analysis, whereas those in the *Elements* depend more on synthesis.) It is so pedantic that it leads Taisbak, in the introduction to his translation, to make a statement whose importance I find quite extraordinary: "When I started to translate the *Data*, I found it very longwinded that a certain phrase kept popping up time and again, several times in every proposition: *if this item is given, that item is also given*; I decided to cancel [*sic*] all those *alsos* and restore them only when they were absolutely necessary. But then I discovered that I was leaving out an essential feature of the *Data*: the Givens hang together in chains, the purpose of any proposition being to produce more links to them" (14; italics in the original.) These "repetitions" are, of course, nothing but *the traces of the underlying X/RCs*, a fact that becomes even more important in the event that Euclid is indeed copying earlier proofs: in this case, it is the best evidence we have that X/RCs were there from the beginning of Greek deductive mathematics. That Taisbak is a self-avowed internalist (8) makes his comment even more striking, for what he confirms, in his gradual realization of the importance of repetitions, is the crucial influence on Greek mathematics of poetic storytelling![78]

The reason why the *Data* give us such a strong argument for the conscious use of X/RC is that in their theorems it is employed even when, quite clearly, it is not necessary for the argument. This is strong proof indeed: if we only found X/RC in well-formed, strong arguments, we might be forgiven for thinking that its use was either unconscious, somehow inherent in the logical process, or superimposed in the editing phase, as mere calligraphy. However, its use when it is not required and when, furthermore, it makes simple, straightforward arguments unnecessarily convoluted, even "totally silly" (the term is Barry Mazur's, commenting on the proof of *Data* 1.5), and needlessly complicated, shows that something deeper is guiding its use. This could only be the conscious application of an established apodeictic technique.

Here is the proof of *Data* 1.5, whose *protasis* reads: "If a given magnitude is subtracted from a given magnitude, the remainder will be given".[79] This is made concrete in a diagram showing the line segment in figure 10.45.

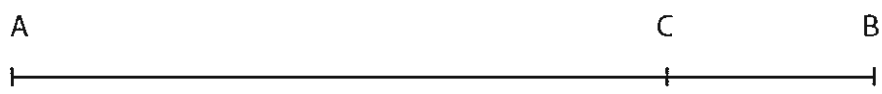

Figure 10.45. The first diagram accompanying proposition 1.5 of Euclid's *Data*.

A proof then follows whose logic is hard to understand by modern standards. I have written

it here in a simple symbolic form for greater clarity, otherwise sticking to both the content and the order of the statements. I use the symbolism $\delta(x)$ to mean that "x is given" (*x dedomenon*) and X:Y, for "X is to Y":

> Let δ(AB:AC). I say that δ(AB:BC). Construct ZD. Since δ(AB:BC), construct ZD:DE equal to it (*Data* 1.2). So δ(ZD:DE). And δ(ZD). Therefore δ(EZ). {EZ is the remainder} (*Data* 1.4). So δ(ZD). Therefore δ(ZD:ZE). Since AB:AC::DZ:DE, [then] turning around (*anastrepsanti*) DZ:ZE::AB:BC. Therefore also δ(AB:BC).

Figure 10.46. The second diagram accompanying proposition 1.5 of Euclid's *Data*.

In other words, this proof depends on the construction of DZ, another segment, on which a point will be placed by applying an earlier proposition, *Data* 1.2, as shown in figure 10.46. This detour, which makes the proof seem unnecessarily convoluted, becomes crystal-clear if we see it as what it really is, the perfect RC we see in figure 10.47. (As always, the construction is excluded from the RC; the pivots are in bold.) There seems to be no other reason for a mathematician to construct a proof such as this other than the simplest one: that that was the way such things were done at that time.

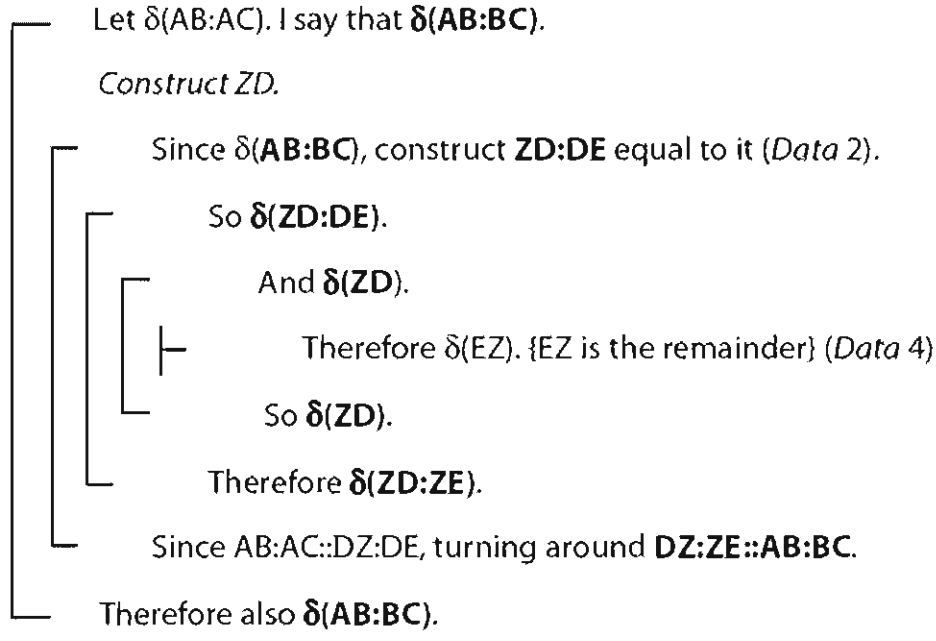

Figure 10.47. The full RC structure of proposition 1.5 of Euclid's *Data*.

6. Conclusion

I have proposed that the methodology of logical demonstration, or proof, was not born *ex nihilo* in Classical Greece, but resulted from the gradual transformation and convergence of various evolving practices, at the heart of which are the cognitive capacities of narrative and the cultural tools of poetry. Using the streetcar analogy, I followed a particular strand of intellectual history from its roots in the cognitive infrastructure and the practice of quotidian *narrative* to its development in the Archaic Age as *poetic storytelling*, to the assimilation of parts of it, in the fifth century BCE, to *rhetoric*, in a methodology for comparing narratives and applying to them

general laws by using certain techniques of poetic storytelling, to the final merging of rhetorical methodology with the world of practical geometry to produce what we know today as mathematical proof, especially as seen in Euclid's *Elements*.

One of the pioneers of the recent multidisciplinary approaches to the study of narrative, the psychologist Jerome Bruner, has stated that narrative is a distinct cognitive mode, irreducible to the bundle of observational, inductive, and deductive practices that are usually packed together as logical, or scientific thinking.[80] "Efforts to reduce one mode to the other," writes Bruner, "inevitably fail to capture the rich diversity of thought" (1985, 11). I fully agree with his comment, yet add the extra clarification, that the impossibility of complete reduction does not rule out the existence of structural similarities and common roots, nor does it make unnecessary the study of the long process of cultural transformation, from narrative roots to their logical progeny, that I have tried to describe.

I believe that the streetcar metaphor is important, especially when first encountering arguments of the relationship of narrative and poetry to logical thinking. The simple assertion that "proof comes from narrative" can sound wild, even absurd. Yet this is probably due to the effect that Sherlock Holmes describes to Watson in the *Adventure of the Dancing Men*: "It is not really difficult to construct a series of inferences, each dependent upon its predecessor and each simple in itself. If, after doing so, one simply knocks out all the central inferences and presents one's audience with the starting-point and the conclusion, one may produce a startling, though possibly a meretricious, effect" (Conan Doyle 1990). My aim was not to startle or to impress, but to substantiate my of the narrative and poetic roots of mathematical proof. I have thus tried to outline "the central inferences," that is, to give some detail to this story of gradual transformation: to describe the main stops of the streetcar named (among other things) Proof. A fuller story remains to be told.

ACKNOWLEDGMENTS

I wish to thank the participants in the July 2005 Mykonos "Mathematics and Narrative" conference, as well as the participants in the July 2007 Delphi "Mathematics and Narrative" conference (the latter of which also led to this volume), for listening to and making valuable comments on early versions of some of the ideas presented here. Thanks also to Petros Dellaportas, Richard McKirahan, Tefcros Michaelidis, and Christos Papadimitriou for listening to early presentations of the ideas on chiasmus and ring-composition and making useful comments. Jean Christianidis, David Herman, Michael Gagarin, Pavlos Kalligas, Dimos Spatharas, and Michalis Sialaros have contributed particularly valuable suggestions, the latter also preventing me from numerous stylistic faux pas, though, yes, of course, any remaining ones are my own. My summer seaside discussions with Barry Mazur, from 2005 to the present, have been a constant, great source of inspiration for applying the investigation of the structural similarities of proofs and stories, more particularly to the early development of Greek mathematics, an inspiration generously interspersed with wonderful insights. Margarita Metzger, my assistant, has been of immense help in this research. But I couldn't have done any of it without the love and support of Dorina, my wife, who kept my soul alive in my long excursion from my more familiar ground of storytelling into the labyrinthine cities of scholarship.

NOTES

1. After reading an earlier draft of this essay, Gagarin (personal communication) had this to say:

> Democracy is a pretty nebulous concept before Cleisthenes in Athens. I prefer to speak in terms of popular participation in decision making, which was widespread in early Greece and even in Homer, where assemblies open to all (it appears) do meet and at least

advise, and where even a Thersites can express an opinion. We cannot be certain what "polis" meant [before Classical times], but it has to be some sort of large group [of] members of the community, however that was defined. This group participated in the making of laws and must, as in Homeric assemblies, have debated, argued, and finally reached some sort of conclusions. [We find this] process at work everywhere in Greece, not just in democracies. . . . This must have involved some of the rhetorical strategies we see in Athens.

2. A more historically informed discussion of this can be found in Doxiadis and Sialaros (2012).

3. The term *rhetoric* as used here refers to a technique that is not limited to the practice of public oratory. This technique, which often extends to the related technique of *dialectic*, is to be found not just in political and forensic speaking but also in the practice of Greek fifth- and fourth-century BCE historiography, philosophy, and medicine.

4. Obviously, there exist also nonverbal narratives. But here I restrict my analysis to narratives expressed in verbal language.

5. I am talking here about the order in which the story is told. The order of the events as they occurred may be different, a distinction narratologists usually make distinguishing between *fabula* (the story as it happened) and *syuzhet* (as it is told).

6. The classic narratological work on this is Bremond (1973), while similar approaches are involved in Labov's work on comparators (1972) and Prince's concept of the disnarrated (2004). An excellent discussion of these notions can also be found in Herman (2002). For a particularly interesting, idiosyncratic view of narrative possibilities, see John Allen Paulos (1998), who discusses narrative possibility from the viewpoint of information theory.

7. Barbara Tversky (2004) has studied the cognitive aspects of a similar kind of modularity in spatial searches, also involving a concept of outlines. For a discussion of some aspects of modularity in both narrative and mathematical thinking and their parallels, see Doxiadis (2004).

8. For theoretical discussions of some of the most unexpected influences of culture on storytelling, such as dance and weaving, see David (2006) and Tuck (2006).

9. For a discussion of the roots of the logical process in Greek legal thought, see the insightful essay of Markus Asper (2004).

10. A famous Homeric example of proverbial wisdom is Hector's response to Polydamas (*Iliad* 12.243). When the latter interprets the fall, from the talons of an eagle, of a wounded serpent into the Trojan camp as an omen from Zeus, Hector disagrees, expressing the *gnômê*-like opinion, "The best omen of all is to defend one's country."

11. The traditional rendition of this concept is "probable." However, David C. Hoffman (2008) has made a strong argument that its original usage could be best rendered by the word "similar."

12. See O'Banion (1992).

13. For the study of narrative as a crucial part of legal thinking, see James B. White (1989), Nancy Pennington and Reid Hastie (1991), Peter Brooks and Paul Gewirtz (1996), and Anthony Amsterdam and Jerome Bruner (2001).

14. Michael Gagarin (personal communication) points out that Antiphon's Second Tetralogy provides a clear illustration of this thesis: both speakers are aware that they are describing the same set of events, but with different narratives, resulting from different interpretations of motives.

15. The locus classicus for this is Stesichorus's *Palinôdia* (ca. 600 BCE). This "second song on the same subject" follows an earlier, nonextant one by the same poet accusing Helen for her crime of instigating the Trojan War. The *Palinôdia* puts forth a revisionist account of the story of Helen, according to which she did not follow Paris to Troy but was saved at the last moment by Aphrodite, goddess of Eros, who took her to safety on Cyprus, sending a phantasm in her place to Troy.

16. In our analysis of Greek rhetoric, we rely both on the theoretical treatises, especially Aristotle's *Rhetoric*, and actual fifth- and fourth-century BCE speeches, mostly by the three earliest, Gorgias, Antiphon, and Lysias. It should be noted, however, that the patterns in the macrostructure of a speech, as well as some of the principles of composition that Aristotle describes, are much more clear-cut and well-defined in the *Rhetoric* than in actual speeches. This discrepancy does not really

affect our overall argument regarding mathematics: since the structural template of mathematical proof we find in Euclid was not really in place before the second half of the fourth century, it makes sense to assume that mathematicians were influenced not just by rhetorical practice but also by the theoretical treatises: it is an accepted fact that Euclid knew his Aristotle and so, it can be assumed, the *Rhetoric* as well. For the discrepancies between rhetorical theory and practice, see in particular Carey (1994, 1996).

17. Outlining of parts of a story can occur throughout a performance, at various levels of organization. For example, Elizabeth Minchin (1992) describes the generic type shared by many Homeric rebukes as *reproach, the problem, generalization, proposal.* The *problem* part is a short narrative, an outline of the situation causing the reproach.

18. Interestingly, the emulation of a similarly stark narrative style has appeared at least two times in the history of twentieth-century literature as a revolutionary advance in style, a breakout from older, more ornate forms of expression: in Hemingway's prose, and in the experiments with "objective writing" of *nouveau roman* writers such as Alain Robbe-Grillet.

19. A modern legal writer repeats this same fact in the context of modern judicial practice, pointing out the simplicity of its style: "Trial stories very much resemble those you learned as a youngster, such as 'Goldilocks'" (Bergman 1989, 12).

20. Among them Hermogenes and Cicero.

21. Viewing the refutation as a separate part of the proof makes more sense when referring to the speech of the defendant: speaking after the accuser, the defendant has already heard the main points of the accuser's attack and can address them one by one. Identifying the refutation as a separate part of the speech makes sense particularly in those trials in which each side is allowed more than one speech, and thus a speaker can refute expressly the points made by his opponent.

22. Johansen (1959) dwells at length on the development of the priamel and other similar paratactic forms in lyric poetry and then fifth-century tragedy, associating these with the development of the use of "general reflection," maxims or *gnômai.*

23. For differences of this notion in Aristotle's *Rhetoric* and actual fifth- and early fourth-century practice, see Carey (1994, 1996).

24. For an extensive discussion of meaning of the notion of *pistis*, see Kennedy (1980, 68–69).

25. Michael Gagarin (written communication) comments thus on this standard definition of a nonartistic *pistis*: apart from laws, all other forms of nonartistic *pisteis*, though nominally "imported" from outside the speech, can actually be as much constructed by the rhetor as the artistic *pisteis* themselves. Witnesses' statements under oath, but also slaves' testimony under torture, were in fact sometimes written in advance by the litigant conducting that particular side of the case. In such cases, then, the nonartistic *pisteis* are also constructed narratives, written by the *logographos* to enrich and support his main narration, with the added advantage of the higher truth value awarded them by the jurors because of their status as nonartistic.

26. Modern scholarship, on the whole, does not accept the older view of Solmsen (1931) that Archaic trials were based on nonartistic *pisteis* alone. Gagarin (1990, 22–32) argues convincingly that some concept of "artistic proof" is present in Greek judicial practice already in Archaic times. Clearly, however, the development of a self-conscious rhetorical art in the fifth century BCE led to a more refined and systematic usage of the artistic *pisteis.*

27. This concept was first formulated by Vygotsky as "mediated action." In mediated action, such as rhetorical practice, "any creativity that occurs involves the transformation of an existing pattern of action, a new use for an old tool." See Wertsch and Tulviste (1996, 69).

28. For good introductions to X/RC in epic, see Notopoulos (1951) and Miller (1982). For RC at the microlevel, see Willcock (1964); for the middle level, see Gaisser (1969) and Reece (1995); for the highest level, see (Beye 1993, 110–11).

29. For the view that X/RC is in some cases unconsciously employed, see in particular the discussions of Minchin (2001) and Nimis (1999) of the extensive Homeric RC analyzed in Stanley (1993).

30. On this, see *The Dance of the Muses* (David 2006).

31. See Willcock (1964), Notopoulos (1951), and Gaisser (1969).

32. Palamedes was a hero, an Achaean fighter in Troy, falsely accused by Odysseus of having accepted a bribe from Priam, in order to betray his side to the Trojans.

33. C. M. Keller and J. D. Keller (1966) give striking examples of such grounding of higher thought procceses on "mere" craftsmanship, in the practice of blacksmiths.

34. Charles H. Kahn (2003, 149–52) mentions detailed technical descriptions of complex buildings, often drafted as the architect was actually working on the construction, as among the first instances of written documents in Greece.

35. The pointing or *deictic* function is crucial to early proof. However, this function is less apparent when proofs are transcribed in the more formal style that becomes prevalent with (and after) Euclid. "This here line" in the oral environment becomes "line *AB*," or simply "*AB*," whereas "*this* is bigger than *that*" becomes "*ABC* is greater than *DEF*." The labeling of points, lines, and shapes by letters is a crucial aspect of Greek written mathematics. But it has less to do with its content and more to do with the translation into writing of oral, live interaction. Obviously, this transition also involves the abandonment of actual (manual) deictic communication.

36. For fascinating group pictures of the development of a "school" and a sense of a tradition in Greek mathematics, see Collins (1998) and Netz (2002).

37. The same, incidentally, as we saw used by the painter of the vase shown in figures 10.22, 10.23, and 10.24.

38. It should be noted that this style, as well as the Euclidean original from which it is translated into a first-person narrative, is a style of mathematical *presentation*: it shows us how a theorem can be proved convincingly, but not how it was originally proved. A narrative describing the process of discovery would of necessity be longer and messier.

39. Of course, we could increase the number of action sentences in this sample, without reducing the others, if we transposed this proof from a Euclidean-type presentation to a live, interactive style of live Greek mathematics, in which many of the state-of-affairs statements can be "proved" at the very moment they are stated, using the compass to measure equality or inequality. So, for example, what is here referred to as a truth proved in proposition 1.20 can be shown by "adding" EB to EF—a simple operation using a compass—and superimposing the "sum" on BF, to show that it is bigger. This would fit our thought experiment much better, but—though it might quite realistically describe the style of a live interaction among early mathematicians—it would not be what we expect from the more formal, Euclidean-style theoretical mathematics.

40. See Netz (1999b). For a discussion of variations, dependent on whether a proposition is a construction or proof, see Doxiadis and Sialaros (2012).

41. The term *diorismos* for this part of the proof is used by Proclus, whereas Heron calls it *prodiorismos*. The word *diorismos* is also used in Greek mathematics as a description of the conditions within which a certain problem is solvable.

42. I am grateful to Michael Gagarin for pointing out that though this clear-cut pattern for the template of the forensic speech is in accord with Aristotle's formalization of it in the *Rhetoric*, it only applies partly to existing fifth- and fourth-century BCE forensic speeches. However, as the form of the mathematical proposition acquired its recognizable form after the *Rhetoric* was written, its determining influence on later practice must be taken into account. It is not a controversial thesis that Euclid read or otherwise knew the work of Aristotle: especially his employment of common notions and postulates has a clear Aristotelian flavor. Thus, it could easily have been the *Rhetoric*, and not the actual rhetorical speeches, that carried more weight with him. (See also note 16.)

43. Though books 7–9 of the *Elements* are mainly concerned with questions that would today be classified as number-theoretic, I shall focus on the geometric propositions for reasons of simplicity.

44. Though we know that some of these are the interpolations of later scholiasts, it is usually assumed that some were there in the original form of the *Elements*.

45. It should be noted that we know from Proclus that at least three earlier mathematicians had written *Stoiheia*, or Elements, i.e., systematic attempts at the presentation of mathematical knowledge. These are Hippocrates of Chios, Leon, and Theudius of Magnesia.

46. Mentioned by Simplicius in his transcription of Eudemus's (non-extant today) *History of Geometry* (ca. 330 BCE) when rendering Hippocrates of Chios's proof of the squaring of the lunes.

47. The formal completeness of syllogisms, in a text addressed to people with some level of expert knowledge, would be "simply a waste of words, because it states much that is obvious" (my translation).

48. Netz (1999a, 214–16) discusses this process, using the term "necessity" instead of "certainty," speaking of the "necessity-preserving properties of Greek mathematical proofs."

49. The tradition of people like Anaximenes and Anaximander, sixth-century BCE architects who were also active in other fields of learning—in effect, "philosophers" in the Greek sense—must also be mentioned. Apart from the fact that, as architects, they undoubtedly dealt in measurements, calculations, and related operations, they were also among the first to record their work in writing, in the form of technical descriptions of their architectural works (Kahn 2003, 149–52).

50. For slight differences in this pattern, depending on whether the theorem is one of construction or proof of a property, see Doxiadis and Sialaros (2012).

51. See (Tuplin 2001) and (Schenkeveld 2002) for arguments that Worthington is finding more X/RC structure than is actually there in the speeches of Dinarchus.

52. This can be seen even more strongly in the *Data* than in the *Elements* (see the discussion in section 5.6).

53. The discovery and classification of all types of X/RC, both at the micro- and mesostructure of Greek proofs, will shed more light on the origins and form of early deductive methodology.

54. The numbered items are outlines of the corresponding segments of the original text. The abridgement, identification of the RC structure, formatting, and numbering is from J. R. Ellis (1991). In this important paper, whose appendices offer a very detailed description of the three-tiered RC construction of the first book of the *History*, Ellis also uses for RC the alternative term *annular structure*.

55. There are strong spatial analogies to this type of thinking, which also connect it wih cognitive aspects of narrative organization (see Tversky 2004; Doxiadis 2004, 2007). There is a tradition in Greek philosophy, in which the road (*hodos*) is a prime metaphor for thinking, including the crucial—to our present argument—concept of crossroads (Szabo 1978; Snell 1978). In fact, the Stoic philosopher and logician Chrysippus ascribes the invention of negative arguments, including the reductio ad absurdum, to the imitation, by early thinkers, of the hound's method of search: when reaching a forking path in search of prey, the hound follows one of the two paths to a certain point and then, if the trail does not continue, he returns to the place of bifurcation and takes the other way (Chrysippus' fragment 7.26, in Von Arnim 1902).

56. This sensorimotor aspect of mathematical activity is strong to this day: a working mathematician's office is rarely—still, in this age of the computer—without a blackboard.

57. Euclid's style is not the style of all Greek mathematics. For variations in later style, see Netz (2007), (2009) as well as Cuomo (2001, 2007).

58. Heraclitus, fragment 93, in Diels, H. and W. Kranz (1966).

59. I have numbered the three statements sequentially, though, as elements of an RC, they might be numbered as 1, 2, and 1*. This is clearly a minimal RC, with "Socrates" as pivot on 1-1*.

60. When there is only a single foil or counterstatement, we call this the reductio ad absurdum.

61. This expansion is a form of complex or tiered RC (see section 5.5.2.5). We shall use the first of the binary Xs of proposition 1.19 as an example of tiering in the next section, in figure 10.44.

62. (Netz 2004) argues that the form in which it has come down to us, i.e., in the record of Simplicius (fifth century CE), of the nonextant account of Eudemus (ca. 330 BCE), the latter written in the ancient (*arhaikon*) style, and appended by Simplicius with some extra things from Euclid, is close enough to Hippocrates' original to allow us to speak of it as our earliest evidence for the methodology of fifth-century BCE mathematics.

63. Hippocrates and Antiphon worked in Athens, as did Gorgias, at least as early as 427 BCE.

64. The fact that a proof in the *arhaikon ethos* is level three, while those in the *Elements* are level four, may also account for the fact that the alternatives are set out in a more orderly way in Euclidean proofs, such as in 1.19.

65. In a series of works, Worthington (1991, 1992, 1994) describes elaborate X/RC structures used by the fourth-century BCE orator Dinarchus.

66. As mentioned in note 51, Worthington's analysis has been contested (MacDowell (1994, 270–71), Schenkeveld (2002, 129), and Tuplin (2001, 389)).

67. For a different discussion of the possible spatial origins of X/RC, see note 55.

68. For a good introduction to what is known of its earliest history, as well as the form that the technique took in Greek and Latin oratory, including a survey of the main ancient sources, see the first chapter in Yates' famous book (1996). For recent evidence that the power of the method can also be explained by the structure of the brain, see Becchetti (2010).

69. This is also the origin of the notion of *topos*, or *locus* in rhetoric.

70. There are obvious connections of such hierarchic thinking with concepts in computer science. The branching off into further layers, each with its own self-contained little proofs, is directly reminiscent of the notion of subroutine. Christos Papadimitriou and Constantinos Daskalakis (personal communication) have noted the similarity of RC to the process of stacking, with its last-in-first-out or, equivalently, first-in-last-out structures.

71. Greek poets used such X/RCs to structure this branching as, ever since the time of Homer, they provided—their other uses apart—the perfect way for handling digressions without losing track of the main narrative (Gaissner 1969).

72. This would happen, as the digression would run, "but we know that . . . (the statement of 1.4), because (the proof of (1.4)), ending in the restatement of (1.4)." This is the exact parallel to the RC assimilating paradeigma in Homer (Willcock 1964).

73. Netz correctly remarks on the repetition of the statement of the *protasis* in the *sumperasma* is a form of ring-composition (2009, 124). That the repetition of the enunciation at the end is "unavoidable" in any form, poetic or otherwise, where "some thesis is to be argued," had already been noted by Fowler (1987, 62), who remarks that the end of argument-type structures naturally repeats "the opening statement as the conclusion (QED)." Netz's other remark on this limited case of RC is that this "ring structure may well be intentional . . . but of course this cannot be proved" (2009, 165). My arguments in this essay, on the prevalence of RC at all levels of Greek theorems, large and small, as well as its essential role in the process of demonstration, go some way toward proving its centrality in Greek mathematics, which also points strongly towards intentionality. Netz's argument for the minimal form of RC he mentions in *Ludic Proof* (2009) is very much that, as he phrases it in one of his chapter's titles, "mathematics turns to literature" in the Hellenistic period mostly for stylistic borrowings that, as a rule, fulfill an aesthetic role. As I discussed at length, I believe this turn occurred much earlier and was crucial to the structure, and not just the aesthetics of deduction. For more on the historical background of the argument, see Doxiadis and Sialaros (2012).

74. I mention the latter case as it is indeed a danger: scholars discovering X/RC constructions where none are, in fact, there. As an example of such scholarly dispute over the existence or not of intricate ring-composition see Nimis (1999) and Minchin (2001), who attack the claims of Stanley (1993) of discovering such patterns in book 18 of the *Iliad*. See also notes 51 and 65, on similar objections to Worthington's work.

75. I am grateful to Barry Mazur, for pointing out the deductive uselessness of the RC in the example from Euclid's *Data* as demonstrating *precisely* the fact that RC must have been, by his time, an ingrained habit of Greek mathematicians.

76. As Taisbak (2003, 13) notes, the *Givens* would be a more correct rendering.

77. For some recent research, see Michalis Sialaros (2011).

78. Another comment which can be made, in the spirit of our previous discussion, is that the proofs in the *Data* are much closer to level five than those of the *Elements*.

79. All the theorems in the *Data* are of the basic form "if x is given, then also y is given." According to Taisbak's illuminating formulation, the term "given" roughly means "that if some items are given, some other items are also given, *into the bargain* so to speak" (2003, 13; italics in original).

80. Bruner (1985), somewhat confusingly, calls this the "paradigmatic mode."

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